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Modal Logics for Continuous Dynamics

J. M. DAVOREN

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MODAL LOGICS FOR CONTINUOUS DYNAMICS

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by
Jennifer M. Davoren
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MODAL LOGICS FOR CONTINUOUS DYNAMICS

Jennifer M. Davoren, Ph.D.

Cornell University 1998

This work is a formal investigation of a number of bimodal and polymodal logics built on a base of propositional **S4**, and is a contribution to the theory of hybrid control systems. It is the first stage of a larger project of developing logics for the design and verification of such systems. A hybrid control system is a network of finite-state digital machines which act on and react to a dynamically changing environment, where such environments may have mixed analog and digital states. Following Nerode, I look to topology to provide a mediating link between the analog and digital worlds; **S4** is taken as a logical foundation since from Tarski and McKinsey, it is the logic of topology.

The base logic **S4F** adds to the $\Box$ (topological interior) of **S4** a modality $[a]$ for representing the effect of an action in an environment; $[a]$ is interpreted by a total function. In this logic, the continuity of a function with respect to a topology is expressible by the scheme: $[a]\Box\varphi \to \Box[a]\varphi$. In the second stage of this study, a fragment of deterministic propositional dynamic logic **DPDL** is overlaid on **S4F** to produce a new modal dynamic logic. In the resulting logic, called **TPDL** (topological propositional dynamic logic), atomic actions are interpreted by continuous functions, and complex actions are formed under the Kleene operations of composition, choice and iteration.

Both a Tarski-style topological semantics and a Kripke semantics are presented for the logics. Building on work of Grzegorczyk, I identify a subclass of topological structures naturally dual to Kripke frames. Topologies in this class are such that every point is contained in a smallest open set. As argued by Nerode, these are precisely the topologies needed to give an account of analog-to-digital conversion.

In addition to Hilbert-style axiomatizations, tableaux proof systems are presented for each of the logics and proved complete. The tableaux completeness proofs construct countable T_0 topologies whose elements are functional terms, in which the term constructor functions are continuous. Finite quotients of the term model are obtained, so establishing the decidability of each of the logics.

Just as this investigation was being completed, the author obtained abstracts of very recent work by Kremer, Mints and Rybakov on "Dynamic Topological Logics" (**DTL**'s), which are **S4**-based propositional dynamic logics. Their logics include a "next" operator corresponding to the $[a]$ modality, for a single atomic action a, and a "star" operator corresponding to $[a^*]$ for atomic a. The abstracts announce axiomatizations of various fragments; for example, the star-free fragment of the logic $\mathbf{DTL}_{\mathcal{H}}$ of homeomorphic functions.

Biographical Sketch

Jennifer Davoren was born in 1963 in Australia, in the town of Griffith, about 700 kilometers west of Sydney. She began studying Mathematics and Philosophy at the University of Melbourne in 1987, graduated with first-class honours in March 1991, and commenced graduate studies in Mathematics at Cornell in August of that year. During 1992-94, she also completed a Masters degree in Computer Science at Cornell. She holds a postdoctoral position at Cornell at the Center for the Foundations of Intelligent Systems.

For Karen

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It was my advisor, Anil Nerode, who first awakened my interest in the ways logic can be a genuinely useful endeavor, rather than merely a pleasant occupation that keeps one off the streets. It was he who wisely insisted that I undertake a master's in computer science concurrent with the Ph.D. in mathematics. Without such sage guidance, I might well have become a recursion theorist.

Anil Nerode has been unfailing in his support of me and my work. The enthusiasm, indeed, exuberance, Anil brings to mathematics is infectious and inspiring. Many was the time I went into Anil's office stuck and discouraged and left buoyed up and full of new ideas. Anil was also able to provide me with important material assistance including an office and the funds to attend conferences.

Some of the foundational ideas of this work emerged in a series of highly useful conversations with Sergei Artemov. I would like to thank Sergei for sharing his insights and his expertise in proof theory. Richard Shore, a minor member of my special committee, gave some astute comments on an earlier draft of this work. These comments prevented at least one error and pointed me in fruitful directions. Special thanks to Robert Constable for being my computer science advisor, and for his comments on the introduction that lead to substantial improvements in it.

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Finally, this dissertation is dedicated to Karen Jones, without whom, not.

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Chapter 1

Modal Logics for Continuous Dynamics

1.1 Introduction

This work is principally a formal investigation of a number of modal logics — bimodal and polymodal logics built on a base of propositional **S4**. These logics have been developed as a contribution to the theory of *hybrid control systems*, and this work is the first stage of a larger project of developing logics for the design, specification, and verification of such systems. Broadly described, a hybrid control system is a network of finite-state digital machines which react to and act on a dynamically changing environment, where such environments may have mixed analog and digital states. Research in the emerging area of hybrid systems is aimed at providing reliable, formally verified, computer control of physical processes, such as aircraft, power grids, and manufacturing facilities.

There is a fundamental tension between the sorts of mathematical structures used to represent analog or physical processes, and the sorts of structures used to represent digital devices; indeed, much of the effort in the research area of hybrid systems lies in addressing this tension.

In the *analog* or "continuous" world view, physical processes are usually modelled as some form of input/output dynamical system, or more abstractly, as a collection of vector fields, where the state space is usually embeddable in a Euclidean space $\mathbb{R}^n$ and so contains continuum-many points; in such models, states evolve on trajectories $x(t)$ changing continuously with (real) time $t \in \mathbb{R}^+$.

In the *digital* or "discrete" world view[1], a digital device or program is usually

[1] I will use the pair "analog/digital" rather than "continuous/discrete" to mark

modelled as some form of finite automaton, with finitely many states and finite input and output alphabets; state transitions are modelled as occurring in discrete steps, so time moments are usually positive integers $n \in \mathbb{N}$.

Following [NK93a], I look to *topology* to provide a mediating link between these two competing world-views. Propositional **S4** is taken as a logical foundation since from McKinsey [McK41] and Tarski and McKinsey [MT44], it is the logic of topology.

1.2 Agents and Actions

The next two sections are devoted to identifying central concepts and laying out a broad framework, with the intention of motivating the logics developed in subsequent chapters. An overview of the logics themselves is given in Section 1.4. The framework given here is loosely based on Nerode and Kohn's "Multiple Agent Hybrid Control Architecture (MAHCA)" [2] [KN92], [NK93b], [NK93c], [KJN+95], [KR97]. It might best be described as an attempt to identify the abstract form of the MAHCA framework, simplified to a single agent.

Starting informally, an *agent*[3] is a "dumb" finite machine operating in and interacting with a complex environment that includes real or "continuous" space and time. Its fundamental operational sequence consists of answering the questions:

- "What is the current state of the environment?", or more simply, "Where am I now?";
- "What state should the environment be in?", or "What is my next goal?"; and

the distinction, reserving the term "continuous" for the property of functions.

[2]The core of Kohn-Nerode MAHCA framework is a procedure which, given a performance specification expressed as an optimization problem of a specific form, extracts an ϵ-optimal feedback control function $\gamma : X \to C$, where X is the state space and $C \subseteq \mathbb{R}^n$ is a space of control values. The procedure draws on differential geometry, calculus of variations, optimal control theory and dynamic programming. This procedure is not discussed here. See also [KNR95], [KNR96b], [KNR96b], [BV97].

[3]Although engaging in a not dissimilar endeavor, the present work does not engage directly with studies in philosophical logic on the logic of (human) action (see, for example, Segerberg's survey [Seg92] and other papers in that issue of *Studia Logica*, or Horty and Belnap's [HB95]) or the AI literature on action and change using the situation calculus and non-monotonic logics (see, for example, Shoham's *Reasoning About Change* [Sho88] and the recent work of Shanahan [Sha97] on the "Frame problem").

- "What action can be taken that will realize the desired goal?", or "What do I need to do to get there?"

and then performing the designated action.

So, an agent acts in an environment in order to bring about a change in that environment. I take a *control agent*[4] to be any finite machine that can:

(a) **sense**: acquire input from its environment, e.g. take sensor readings of various components of the state of its environment.

(b) **act**: effect change in its environment by performing one, or a sequence, of its primitive actions.

(c) **convert data**:

  (i) convert sense data into digital form, suitable as input for internal finite automata, and

  (ii) translate symbolic action instructions, which are digital output from an internal finite automaton, into action.

(d) **use knowledge**: access a "knowledge base" which includes symbolic descriptions of the known or predicted effects of its actions on the environment, say of the form:

> if the current state is in region A and action a is performed
> then the resulting state is in region B

where A and B are symbolically described *regions* or *sets of states* of the environment, and a is a symbolic representation of an action.

(e) **plan**: formulate goals, where a goal is a symbolically described region G of the environment. The planning module is some form of finite state automaton, internal to the agent, which takes as input digitalized sense data i and utilizes the knowledge base to produce a symbolic G as output.

(f) **compute**: given as input digitalized sense data i and a goal G, and utilizing the knowledge base, determine by finite computation whether there is an action it could perform that would realize the given goal; if so, output the symbolic action instruction a for that action. This module is also some form of finite state automaton.

[4]The properties listed here are implicit in the description of the MAHCA "agent controllers" in [KN92], [NK93b], [NK93c], [KJN+95], [KR97].

(g) **adapt**: whenever a computation of type (f) fails, report the failure to the planning module, which reformulates the goal; failure may also be reported to the knowledge base if the language is sufficiently expressive.

In looking for *logics* appropriate for such agents, I take the objective to be two-fold:

(1.) Identify formal languages and logics suitable for the "knowledge base" of an agent, so that the computation in (f) might be assisted by an internal on-line automated theorem prover, as in the MAHCA architecture. If G is the symbolically described goal state, and from digitalized sense data input i, it is determined that C symbolically describes the current region, then the computation in (f) has to *find* a symbolic action instruction a such that the sentence:

> if the current state is in region C and action a is performed
> then the resulting state is in region G

is *provable from the knowledge base*, as formalized in some language and logic.

(2.) Identify formal languages and logics suitable for (external or off-line) reasoning *about* the behavior of an agent and its interactions with the environment, for the use in the formal specification and verification of such systems.

The modal logics investigated in this work go some way towards both (1.) and (2.). Regions of the state space are denoted by modal propositional formulas, and the language includes "action modalities" $[a]$, so that the formula:

$$C \to [a]G$$

means:

> if C, then action a will always make it the case that G

But being propositional logics, they are limited in their expressive power. In first-order extensions of these logics, one would have a richer vocabulary with which to describe regions of the environment. A further stage in this project is to investigate decidable fragments of such first-order extensions.

Having outlined the sort of entity an *agent* might be, what is an *action*? Starting naively, and taking the simplest case first, I think of an *action* as anything an agent can do whose *effect* can be modelled *deterministically* as a *total function* $f : X \to X$, where the state space X is a representation of the agent's environment.

For example, an agent might be part of an automated control system in a manufacturing plant, say a machine that supervises a tank of liquid chemicals. Points

$x = (x_1, ..., x_n) \in X$ might include coordinates x_i for the liquid volume, temperature, and concentration of various chemicals in a tank. Global or *real time* $t \in \mathbb{R}^+$ is likely to be a distinguished coordinate of the state space X; perhaps also a relative time or "clock" coordinate such as the time since a particular event took place. In addition to real valued coordinates x_i, there could also be discrete valued coordinates representing data like whether a particular switch is on or off[5]. The state space X could also be expanded to include coordinates for the agent's own internal states, since the agent itself *lives in* its environment. For an agent in such a system, an action might be adding a certain quantity of a chemical to a tank, physically brought about by sending electrical (analog) signals which activate various mechanical devices (actuators). The mathematical representation $f(x)$ of the effect of this action on a state x might be a prediction of what the volume, temperature and chemical concentrations etc. will be 60 seconds *later*, with the global time coordinate incremented by 60 seconds, assuming it takes say 15 seconds for the chemicals to actually get in the tank.

The mathematics involved in modelling and predicting the effect of an action in a physical system may draw on work in differential equations, functional analysis, calculus of variations, dynamical systems, differential geometry, linear and nonlinear systems theory, and whatever else is useful. In a mathematical model of the effect of an action, call it $\mathfrak{M}_{\text{analog}}$, the state space X representing the environment will usually be imbued with a great deal of rich mathematical structure.

In the paradigmatic case, $\mathfrak{M}_{\text{analog}}$ imbues X with the structure of a C^∞ or C^r differentiable manifold and includes a coordinate for positive real time, say $X = Y \times \mathbb{R}^+$. In this case, the function $f : X \to X$ is obtained from the *flow* $F : Y \times \mathbb{R}^+ \to Y$ of a *vector field* v on Y. A vector field defines a system of differential equations, which in favorable circumstances has a unique solution for each initial condition; the flow is the family of solutions or trajectories $y(t)$ expressed as a function of initial conditions and time. Flows can also be written as a family $\{F_t\}_{t \in \mathbb{R}^+}$ of *evolution operators* $F_t : Y \to Y$ given by $F_t(y) = F(y, t)$, which represent how a point $y \in Y$ (i.e. *without* the temporal coordinate) will evolve according to vector field v over a time interval of *duration* t. The evolution operators satisfy the semigroup axioms:

$$F_0 = \mathbf{1}_Y \quad \text{and} \quad F_s \circ F_t = F_{t+s}$$

for $t, s \in \mathbb{R}^+$, where $\mathbf{1}_Y$ is the identity function on Y. These equations are also known as the Chapman-Kolmogorov laws, and are taken to express a principle of *determinism* ([AMR83], §4.1).

In this setting, a *primitive action* might be represented as a *switching of vector fields*. Let v, u be vector fields on Y, with flows $F, G : Y \times \mathbb{R}^+ \to Y$, respectively.

[5]See [KNR96b] for a discussion of ways "continualizing" digital states so that they may be treated on an equal footing with real-valued ananlog states.

Causing some event to occur t units of time hence, then allowing the process to evolve for a further s units of time, amounts to:

> the process evolving according to vector field v for time duration t, then being switched to vector field u for a further time duration s.

Then the effect on a point $x = \langle y, \tau \rangle$ is given by:

$$f(x) = f(y, \tau) = \langle (G_s \circ F_t)(y), \tau + t + s \rangle$$

(In the chemical tank example, $t = 15$ and $s = 45$ seconds.)[6] Alternatively, a primitive action might amount to *choosing to stay on the same vector field.* In that case, we have $u = v$ and $G = F$, so the effect on a point $x = \langle y, \tau \rangle$ is given by:

$$f(x) = f(y, \tau) = \langle (F_s \circ F_t)(y), \tau + t + s \rangle = \langle F_{t+s}(y), \tau + t + s \rangle$$

Clearly, the effect of a single action can admit many *different* representations f: one can keep the same pair of vector fields but vary the time durations t and s, or one may refine the modelling to produce a new pair of vector fields. For now, something gets to be called an action if someone, somewhere, can represent it as one or more systems of differential equations with unique solutions, solve those equations and report back, preferably with a nice computable formula for f.

To obtain a deterministic representation of the effect of an action, many simplifying assumptions have to be built in to the model $\mathfrak{M}_{\text{analog}}$[7]. There are inevitably some factors in the dynamics of the situation that have to be ignored for the purposes of modelling; the world (environment) is invariably more complicated than any mathematical model of it. In particular, deterministic models must ignore the *imprecision* with which an agent *interacts* with the world. Such modelling will have to assume that an agent's *actuators* behave perfectly and are perfectly precise; for example, that precisely g grams of some chemical is added to a certain location in the tank precisely t seconds after a signal is received. Such models also leave no room to talk about the precision of an agent's *sensors* since the modelling is based on the mathematical representation of a point x in the state space, not on any sensor reading of x.

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[6]The "switching vector fields" idea of action seems not incompatible with von Wright's conception of action in his *Causality and Determinism* [vW74] (quoted in [Seg92]): "To act is to interfere with the course of the world, thereby making true something which would not otherwise (i.e. had it not been for this interference) come to be true of the world at that stage of its history." (p.39)

[7]Simplifying assumptions are also required for a non-deterministic modelling as a set-valued function $f : X \to \mathcal{P}(X)$, though usually fewer of them.

1.3 Topologies and Continuity

One of the novel ideas in [NK93a] is to use a *topology* on the state space to reflect the imprecision with which an agent interacts with the world, and more generally, to provide a layer of meaning which mediates between the analog and digital world views.

Formally, a *topology* $\mathcal{T}$ on a space X is any family of subsets of X that is closed under finite intersections and arbitrary unions, and contains the empty set $\emptyset$ and the whole space X. Sets $U \in \mathcal{T}$ are called *open* (relative to $\mathcal{T}$).

Topologies are often represented by a family $\mathcal{B}_{\mathcal{T}} \subseteq \mathcal{T}$ of *basic open sets*, with the property that whenever U is open and $x \in U$, there is a basic open set $B \in \mathcal{B}_{\mathcal{T}}$ such that $x \in B$ and $B \subseteq U$. So the basic open sets are the *small* open sets, and every open set is a union of basic open sets.

Think of the basic open sets of a topology $\mathcal{T}$ on a state space X as the *simple meaningful regions* of X. Such sets act as a collection of lenses or filters which mediate between an agent and its environment, as represented by X. Thought of as lenses, the basic open sets reflect the *precision* with which points in X can be discriminated. Thought of as filters, the basic open sets are what an agent uses to *make sense of*, or *give meaning to*, continuum-much information.

For example, the standard topology $\mathcal{T}_{\mathbb{R}^n}$ on Euclidean space $\mathbb{R}^n$ (or more generally, any metric space) has as a basis the collection of all balls $B(x, \epsilon)$ — the points y of distance less than ϵ from x — for all x and all real numbers $\epsilon > 0$. This means that any two distinct points x and y can be distinguished by disjoint balls $B(x, \epsilon)$ and $B(y, \epsilon)$ by taking ϵ less than half the distance them, *no matter how small that is*, and for any ball around x, there is a yet *smaller* ball inside it. So $\mathcal{T}_{\mathbb{R}^n}$ can be thought of as a topology of *perfect* or *infinite precision*, generated by uncountably many "meaningful regions" $B(x, \epsilon)$ shrinking down to a point x.

In the models $\mathfrak{M}_{\mathsf{analog}}$, where X is a C^∞ or C^r differentiable manifold, the topology $\mathcal{T}_{\mathsf{analog}}$ is inherited from the differential structure; X is locally "identical" (homeomorphic) with an open subset of $\mathbb{R}^n$ for some n, so $\mathcal{T}_{\mathsf{analog}}$ will be a perfect precision topology.

On the other hand, for any *physically realizable* agent, there are intrinsic limits to the precision with which it can interact with its environment. Perfect precision is not implementable, and single points in space and time are not physically meaningful. For example, no physically realizable sensor can discriminate between points whose distance apart is smaller than the altitude of light waves. Likewise, no physically realizable clock can discriminate between time instances closer together than, say, the period of the harmonic oscillation of an electron in a helium atom. In the world as we know it, there are smallest discernible regions of space and time. For any particular agent and its environment, there will always be *smallest meaningful*

quantities, of temperature or volume or whatever, which mark the limits of its powers of discrimination.

Call a topology $\mathcal{T}$ on X a *digital topology* or D-*topology* if every point x is contained in a *smallest* open set (relative to $\mathcal{T}$)[8].

So $\mathcal{T}$ is a D-topology on X exactly when, for each $x \in X$, the intersection of all open sets containing x:

$$B_x = \bigcap\{U \in \mathcal{T} \mid x \in U\}$$

is itself open, and so is the *smallest* open set containing x. The D-topology condition is the requirement that any descending chain of smaller and smaller open sets containing a point x must eventually *stop*, marking the limits of discrimination, and the region B_x at which this descent stops is the *smallest meaningful region* containing the point x. It is readily shown that the collection of all such B_x's forms a basis for the topology $\mathcal{T}$. Points x and y are *indistinguishable* through the lenses of $\mathcal{T}$, or *have the same meaning* relative to $\mathcal{T}$, exactly when x and y share the same basic open set in $\mathcal{T}$; i.e. $B_x = B_y$.

From within the world view of analog mathematics, D-topologies may seem quite bizarre; they lack the "separation" properties that are taken for granted from Chapter 2 onwards of most texts in analysis or topology or their applications. A D-space $(X, \mathcal{T})$ is Hausdorff or even T_1 only in the trivial case when the topology $\mathcal{T}$ is discrete (i.e. *every* subset of X is open, and supposedly "meaningful".)

But in the digital world view, D-topologies are just the trick. A D-topology on an state space X with continuum-many points is a set of lenses through which one can get a *digital* view of an *analog* world, or a set of filters in virtue of which a "dumb" agent can make sense of continuum-much information.

From [NK93a], a D-topology $\mathcal{T}$ on X naturally defines an *analog-to-digital* conversion. Let $\{B_i\}_{i \in I}$ be the collection of distinct sets B_i such that $B_i = B_x = \bigcap\{U \in$

[8]These topologies are identified in [NK93a], §5.2, by the name "AD-topologies" to emphasize that they are topologies suitable for describing analog-to-digital conversion. There, the term "small topology" is also used to refer to any *subtopology* of the standard topology on a state space. Any *finite* small topology is a D-topology.

D-topologies were first identified by Grzegorczyk in [Grz67], where they are given the name "totally distributive" topologies. There, the defining property is:

$$cl_{\mathcal{T}}(A) = \bigcup\nolimits_{x \in A} cl_{\mathcal{T}}(\{x\})$$

for all $A \subseteq X$. This is equivalent to the property that an arbitrary union of closed sets is closed, or dually, an arbitrary intersection of open sets is open; these properties are in turn equivalent to the defining property of D-topologies.

$\mathcal{T} \mid x \in U\}$ for some $x \in X$, where I is a (possibly infinite) discrete index set. Then define a map $AD : X \to I$ by $AD(x) = i$ iff $B_i = B_x$. So AD identifies points that belong to the *same smallest* open set in $\mathcal{T}$, and hence belong to all the same open sets in $\mathcal{T}$. When I is finite, the AD map models the conversion of sense data into digital data suitable as input for internal finite automata, as required in the description of an agent. Mathematically, the AD map is just the Stone T_0 quotient map; we return to this point in Section 3.3.

For a given agent with a set of primitive actions denoted by $\{a_j\}_{j\in J}$, let

$$\mathfrak{M}_{\textbf{analog}} = (X, \mathcal{T}_{\textbf{analog}}, \{f_j\}_{j\in J}, +\text{more structure})$$

denote the mathematical structure in which the functions $f_j : X \to X$ come from the flows of pairs of vector fields associated with actions a_j. Now take a D-topology $\mathcal{T}_{\textbf{digital}}$ which is a *subtopology* of $\mathcal{T}_{\textbf{analog}}$; i.e. $\mathcal{T}_{\textbf{digital}}$ contains only *some* of the open sets of $\mathcal{T}_{\textbf{analog}}$, but is closed under arbitrary intersections (as well as arbitrary unions). Structures of the form:

$$\mathfrak{M}_{\textbf{digital}} = (X, \mathcal{T}_{\textbf{digital}}, \{f_j\}_{j\in J})$$

are the ones to keep in mind for the logics subsequently developed in this investigation.

How might one *find* a D-topology on X? One sure way is to start with a *finite* open cover $X = \bigcup_{k<m} U_k$ of sets U_k open in the original topology $\mathcal{T}_{\textbf{analog}}$, then let $\mathcal{T}_{\textbf{digital}}$ be the topology obtained by taking all (finite) unions and intersections of the U_k's. The basic open sets will be those sets B_i that are *join-irreducible* in $\mathcal{T}_{\textbf{digital}}$, considered as a lattice of sets; i.e. with the property that whenever $B_i \subseteq U \cup V$ then either $B_i \subseteq U$ or $B_i \subseteq V$. Such a $\mathcal{T}_{\textbf{digital}}$ will be a *finite* topology; i.e. the total number of open sets is finite, and all finite topologies are D-topologies.

If X is *compact* in the original topology $\mathcal{T}_{\textbf{analog}}$, then we at least know *there is* a finite open cover $X = \bigcup_{k<m} U_k$.

A harder question: How might one go about finding an open cover, and thence a D-topology, which directly encodes a particular agent's (current) level of imprecision, and includes regions of the state space that are appropriately *meaningful* for the agent, and open in the original topology $\mathcal{T}_{\textbf{analog}}$?

Suppose the coordinates of points $x \in X = Y \times T$ in the state space are $x = (x_1, ..., x_n, t)$, with time $t \in T \subseteq \mathbb{R}$. As a start, identify a *precision limit* $\delta_i > 0$ for each non-temporal real-valued coordinate x_i; i.e. differences between values x_i and x_i' smaller than δ_i are *not* meaningful for the agent. Use the δ_i as the measure of the smallest open set in the projection X_i of X onto its i^{th} coordinate, so any interval must be of length at least δ_i. Then try to identify critical or threshold values c which when detected by the agent should instigate action, and add intervals, say of the form:

$$(c - \delta_i, c + \delta_i) = \{x_i \in X_i \mid c - \delta_i < x_i < c + \delta_i\}$$

(of length $2\delta_i$, just for good measure). For example, x_i is temperature and $c = 100°$ Celsius so water is boiling. A meaningful region of X_i might be any subset that can be defined by a finite set of *strict inequalities* or *constraints*[9], subject to the requirement that the intersection of any collection of meaningful regions is of size *at least* δ_i, so still meaningful. Digital coordinates x_i, say $x_i \in \{0, 1\}$, can be embedded into an interval in the reals and treated similarly. Say treat x_i as a value in the open real interval $(0, 1)$ and cover it with $(0, \frac{1}{3})$ and $(\frac{2}{3}, 1)$ together with the whole interval $(0, 1)$. A finite cover of the *non-temporal* part Y of the state space might then be pieced together from finite covers of its coordinate projections. For the time coordinate $t \in T$, one may want to proceed differently[10]. One should still identify a precision limit $\delta > 0$, but instead of a *finite* open cover, it may make more sense to take a *countable* open cover of intervals with intersections of at least length δ. This still gives a D-topology, and the $\delta > 0$ lower bound ensures that any sequence $\{t_k\}$ of timings associated with a sequence of actions is *non-Zeno* or *realizable*[11], meaning only a finite number of actions can occur in a finite interval of time. The task of systematically generating "meaningful" open covers is a worthy object of further investigation, but it is not pursued here.

The last in my shopping list of concepts is *continuity*. Informally, a function f on a space X is continuous if a small variation between x and y gives rise to only a small variation between $f(x)$ and $f(y)$. Thinking of f as the effect of an action, continuity seems like a most pleasant and desirable property: the action doesn't give rise to big "jumps" or "gaps". Of course, a formal account of "small variation" requires reference to a topology.

Formally, a function $f : X \to X$ is *continuous* with respect to a topology $\mathcal{T}$ on X if whenever a set U is open, then the set of points x which get mapped by f into U is also open, relative to $\mathcal{T}$; more succinctly, "the inverse-image under f of an open set is open", where the inverse-image is:

$$f^{-1}(U) = \{x \mid f(x) \in U\}$$

If the open sets in $\mathcal{T}$ are the *meaningful* or *discernible* regions of the state space, then the continuity of f means that whenever U is meaningful or discernible, then so is the set $f^{-1}(U)$ of points which get mapped by f into U.

In the model $\mathfrak{M}_{\text{analog}}$ above, the functions $f_j : X \to X$ are obtained from compositions of evolution operators of flows of vector fields. Under suitable hypotheses, a

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[9] So we will need a first-order logic if we are to define such regions *within* the formal language.

[10] The issue of time as a distinguished variable will be addressed in future investigations.

[11] The notion of a realizable sequence of times is defined in [NK93a], §2.

C^r vector field has a C^r flow[12], so the f_j are always continuous with respect to the original topology $\mathcal{T}_{\textsf{analog}}$ (although their derivatives are likely to be discontinuous at points where the vector field switches). Note, however, that even when $\mathcal{T}_{\textsf{digital}}$ is a subtopology of $\mathcal{T}_{\textsf{analog}}$, the continuity of f_j w.r.t. the larger topology $\mathcal{T}_{\textsf{analog}}$ implies *nothing* about the continuity of f_j w.r.t. the smaller topology $\mathcal{T}_{\textsf{digital}}$. (And conversely, continuity w.r.t. a subtopology implies nothing about continuity w.r.t. the full topology.)

In the case when $\mathcal{T}_{\textsf{digital}}$ is a *finite* topology, say from a finite open cover in $\mathcal{T}_{\textsf{analog}}$, with basic open sets $\{B_i\}_{i<n}$, we have for each $i < n$,

$$f^{-1}(B_i) = \bigcup\nolimits_{j \in J_i} B_j$$

for some index set $J_i \subseteq \{0, \dots, n-1\}$, since $f^{-1}(B_i)$ being open is a union of basic open sets in $\mathcal{T}_{\textsf{digital}}$. This means one can write out all the basic inclusion relations, for each $i < n$ and $j \in J_i$,

$$B_j \subseteq f^{-1}(B_i)$$

i.e.

$$\text{if } \; x \in B_j \; \text{ then } \; f(x) \in B_i$$

which completely map out the behavior of f on open sets. When f represents the effect of an action a, this translates as:

if B_j then action a will always make it the case that B_i

A symbolic representation of such inclusions is the sort of thing that should be found in the knowledge base of an agent.

More generally, when f is continuous with respect to any D-topology $\mathcal{T}_{\textsf{digital}}$ on X, then whenever x and y share the *same* smallest open set, and so are indistinguishable through the lenses of $\mathcal{T}_{\textsf{digital}}$, or have the same meaning relative to $\mathcal{T}_{\textsf{digital}}$, their images $f(x)$ and $f(y)$ will also share the *same* smallest open set and so be indistinguishable relative to $\mathcal{T}_{\textsf{digital}}$. So the continuity of f with respect to $\mathcal{T}_{\textsf{digital}}$ means that the action represented by f *respects* or *preserves* the precision limitations and meanings of the agent, as those limitations and meanings are reflected in $\mathcal{T}_{\textsf{digital}}$.

So as argued in [NK93a], continuity of f with respect to a suitable D-topology can be construed as a *performance specification*. To formally *verify* that such a specification is satisfied, we need a logic in which the purely topological notion of continuity is expressible.

[12] A C^r function is continuous, and for $1 \leq n \leq r$, its n^{th} derivative is also continuous.

1.4 Overview of the Logics

Starting from first principles, I begin by looking for the simplest logic available in which the purely topological notion of continuity is expressible. From the definition: "the inverse-image of an open set is open", the two ingredients are *open sets* and *functions*, so both must be expressible in the logic. The investigation starts with a bimodal logic **S4F** combining two "off-the-shelf" logics: propositional **S4** and a modal logic known as **KF** ("F" for "function").

From McKinsey and Tarski in [McK41] and [MT44], the **S4** axioms for $\Box$, namely:

$$\begin{array}{ll} \Box\mathbf{K}: & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ \Box\mathbf{T}: & \Box\varphi \to \varphi \\ \Box\mathbf{4}: & \Box\varphi \to \Box\Box\varphi \end{array}$$

characterize the interior operator of a topology, and dually, the **S4** $\Diamond$ corresponds to the closure operator. A set is open in a topology when it is equal to its own interior, so "φ is open" is expressed by the formula $\varphi \leftrightarrow \Box\varphi$. In the relational Kripke semantics [Kri63], **S4** is the logic of reflexive and transitive binary relations. Continuing the metaphors of the basic open sets of a topology as a set of lenses or filters, the formula $\Box\varphi$ might be read as "discernibly φ" or "meaningfully φ", since

$$\|\Box\varphi\| \doteq int_{\mathcal{T}}(\|\varphi\|)$$

is the union of all basic open sets contained in $\|\varphi\|$.

The axioms for the "Box" modality of **KF** — here $[a]$ for "action" — are:

$$\begin{array}{ll} [a]\mathbf{K}: & [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi) \\ [a]\mathbf{F}: & [a]\varphi \leftrightarrow \langle a\rangle\varphi \end{array}$$

where $\langle a\rangle$ is defined as $\neg[a]\neg$, and the logic is closed under the rule of $[a]$-necessitation. The logic can be found in [Lem77][13], where it is identified as characteristic for total (serial) and functional (deterministic) binary relations in the Kripke semantics. In a sense, the $[a]$ operator is nothing more than the "next-time" or "next-state" modality of temporal logics[14], given a more abstract semantics. The novelty here lies

[13]The source manuscript of the "Lemmon Notes" [Lem77] is dated 1966, and was a collaboration of E. J. Lemmon and Dana Scott. It was edited for [Lem77] by Krister Segerberg.

[14]The first appearance of the **F** axioms seems to be in A. N. Prior's [Pri57] as the axioms for the "tomorrow it will be the case that" modality, and appear again in that guise in [Seg67]. See also G. H. von Wright's work on the "And Next" modality [vW65] and Appendix B of Prior's [Pri67].

in combining it with the **S4** $\Box$ and $\Diamond$ modalities to give symbolic representation to a topology as well as an arbitrary function. In the topological semantics, the **F** axioms for $[a]$ are satisfied by the inverse-image of a total function, with

$$\|[a]\varphi\| \doteq f^{-1}(\|\varphi\|)$$

and in the logic **F**, it is provable that $[a]$ commutes with all Boolean operations[15].

The formula $[a]\varphi$ can be read as "action a always makes it the case that φ"[16]. Formulas of the form:

$$\psi \rightarrow [a]\varphi$$

then read: "whenever ψ, then action a always makes it the case that φ" or more succinctly, "action a always takes ψ states to φ states". Such a formula is true (evaluates as the whole space) in a topological model $\mathfrak{T} = (X, \mathcal{T}, f; \xi)$ exactly when, for all $x \in X$:

$$x \in \|\psi\|_\xi \quad \text{implies} \quad f(x) \in \|\varphi\|_\xi$$

where ξ is a valuation of atomic propositions as subsets of X. More generally,

$$\psi \rightarrow [a]^k\varphi$$

reads "k iterations of action a always takes ψ states to φ states", where $[a]^0\varphi$ is just φ and $[a]^{k+1}\varphi$ is $[a][a]^k\varphi$. Formulas of this form express the sort of basic facts that should be found in the knowledge base of an agent. If φ is a goal region generated by an agent's planning automaton, and ψ is a region containing the current state, then the internal control automaton should be looking for an action a such that:

$$\psi \rightarrow [a]\Box\varphi$$

is provable from the agent's knowledge base[17].

In the bimodal language $\mathcal{L}_{\Box a}$, the property of continuity is expressible by the axiom scheme:

$$\textbf{Cont}: \quad [a]\Box\varphi \rightarrow \Box[a]\varphi$$

[15]The $[a]$**F** axiom scheme says that $[a]$ commutes with negation, and $[a]$**K** together with its provable converse say that $[a]$ commutes with (classical) implication.

[16]Dually, $\langle a\rangle\varphi$ would be read as "action a sometimes makes it the case that φ". Axiom $[a]$**F** says that, for *primitive* or *atomic* actions, "sometimes" is the same as "always".

[17]So a richer language and logic in which to talk about *complex* actions might be quite useful.

In the topological semantics, the scheme translates as:

$$f^{-1}(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(f^{-1}(A)) \quad \text{for all subsets } A \subseteq X$$

and this condition is satisfied exactly when the function f (interpreting actio. a) is continuous with respect to the topology $\mathcal{T}$ on X [18]. The logic obtained from **S4F** by the addition of the **Cont** scheme is given the name **S4C**: the propositional bimodal logic of one continuous function.

In examining the relationship between the topological and Kripke semantics, I look to Grzegorczyk [Grz67], and back to McKinsey and Tarski [MT46], with some assistance from Scott [Sco72]. Start by defining a reflexive and transitive relation $R_{\mathcal{T}}$ on X from a topology $\mathcal{T}$ on X by[19]:

$$(x,y) \in R_{\mathcal{T}} \quad \text{iff} \quad \text{for all } U \in \mathcal{T},\ \ x \in U \text{ implies } y \in U$$

Thinking of a topology $\mathcal{T}$ as a collection of meaningful regions of X, $(x,y) \in R_{\mathcal{T}}$ says that y has all the same (topological) meaning as x has; $R_{\mathcal{T}}$ is the *meaning relation* of $\mathcal{T}$. So x and y will have the *same* meaning when both $(x,y) \in R_{\mathcal{T}}$ and $(x,y) \in R_{\mathcal{T}}$[20]. To go the other way, start with an **S4** Kripke frame (W, R), and define a topology $\mathcal{T}_R$ on W by taking as basic open sets all the "upper cones" under R [21]:

$$B_w = \{v \in W \mid (w,v) \in R\}$$

[18]The **Cont** scheme might be read as: "if action a will make it the case that *discernibly* φ, then *discernibly*, action a will make it the case that φ". Or try replacing "discernibly" with "meaningfully".

[19]In [Grz67], the relation is defined by:

$$(x,y) \in R_{\mathcal{T}} \quad \text{iff} \quad x \in cl_{\mathcal{T}}(\{y\})$$

which is provably equivalent to the given definition, taken from [Sco72].

[20]Identifying all such points x and y gives the Stone T_0 quotient of $(X, \mathcal{T})$, with the quotient map the AD converter map discussed above.

[21]The topology $\mathcal{T}_R$ is known as the "Alexandroff topology" ([Joh82], [Smy92]) when R is a partial order, and also goes by the name "cone topology" [Mi95]. In [Grz67], the equivalent topology $\mathcal{T}$ on W defined from R by:

$$cl_{\mathcal{T}}(A) = \{w \in W \mid (\exists v \in W)[\,(w,v) \in R \text{ and } v \in A\,]\}$$

for all $A \subseteq W$, is attributed to [MT46]. This last equation can also be obtained from Jónsson and Tarski's work on Boolean algebra with operators [JT51].

For all reflexive-transitive relations R, the induced topology $\mathcal{T}_R$ is a D-topology, and if $\mathcal{T}$ is a D-topology on X then $\mathcal{T}_{R_\mathcal{T}} = \mathcal{T}$; i.e. the topology induced by $R_\mathcal{T}$ is $\mathcal{T}$ itself. So D-spaces are the natural topological duals of **S4** Kripke frames. Indeed, this exact correspondence can be recovered from Grzegorczyk's [Grz67].

The transformations between topologies and reflexive-transitive relations also reveal an elegant characterization of continuity in the Kripke semantics. In one direction, if $\mathfrak{X} = (X, \mathcal{T}, f)$ is a topological structure for $\mathcal{L}_{\Box a}$, with $R_\mathcal{T}$ the induced binary relation on X, then the continuity of f w.r.t. $\mathcal{T}$ implies f is *monotone* w.r.t. $R_\mathcal{T}$. This is a variant on the theme of "continuity implies monotonicity" in cpo and domain theory. What is quite pleasing is the other direction: if $\mathcal{K} = (W, R, F)$ is a Kripke frame for $\mathcal{L}_{\Box a}$, with $\mathcal{T}_R$ its induced topology (and a D-topology) on W, then the R-monotonicity of F implies that F is continuous w.r.t. $\mathcal{T}_R$. So here, "continuity *equals* monotonicity".

The "continuous dynamics" in the title of this work ambiguously refers to both the dynamics of analog or "continuous" processes, and the enterprise of putting a "continuous" spin on *dynamic logic.* In the second stage of the project, propositional dynamic logic **PDL** ([FL79], [Pra79], [Par81], [Seg82], [BHP82]) is overlaid on **S4C** to form a new *modal-based* propositional dynamic logic.

Atomic actions $a \in \Sigma$ are interpreted by continuous total functions, and compound actions α are generated from Σ by the Kleene operations of composition, sum (nondeterministic choice) and iteration (star)[22]. The resulting logic is given the name **TPDL**, *topological propositional dynamic logic.* The modalities $[\alpha]$ and $\langle\alpha\rangle$ remain equivalent if $\alpha = a_1 \cdots a_n$ is a simple composition of atomic actions, but of course they diverge in the presence of sum and iteration. The formula $[\alpha]\varphi$ can be read as "action α *always* makes it the case that φ", while $\langle\alpha\rangle\varphi$ is read as "action α *sometimes* makes it the case that φ".

In the topological semantics for **TPDL**, the modalities $\langle\alpha\rangle$ and $[\alpha]$ for compound actions α are interpreted by unary operators $\sigma(\alpha)$ and $\pi(\alpha)$, respectively, on the power set $\mathcal{P}(X)$. These operators are generated from the inverse-image operators $\sigma(a) = \pi(a) = f_a^{-1}$ of a family of functions $f_a : X \to X$ for atomic actions $a \in \Sigma$,

[22]The "test" operation is omitted at this stage, pending a further clarification of an appropriate semantics. So what is overlaid on **S4C** is actually the test-free fragment of *deterministic propositional dynamic logic* **DPDL**, further restricted to atomic actions whose interpretations are both functional (deterministic) and *total* relations. **DPDL** is studied in [BHP82]; its precursor can be found in a programming logic of [Con77], where atomic commands are also interpreted by partial functions. Within the "algorithmic logic" school of Salwicki and Mirkowska, the logic of deterministic total actions is briefly studied in [MS87], Chp. V, §8.

where each f_a is continuous w.r.t. a topology $\mathcal{T}$ on X. The modalities are given by:

$$\|\langle\alpha\rangle\varphi\| = \sigma(\alpha)\left(\|\varphi\|\right) \quad \text{and} \quad \|[\alpha]\varphi\| = \pi(\alpha)\left(\|\varphi\|\right)$$

where, as one would expect,

$$\begin{aligned}
\sigma(\alpha\beta)(A) &\doteq (\sigma(\alpha) \circ \sigma(\beta))\,(A) \\
\sigma(\alpha+\beta)(A) &\doteq \sigma(\alpha)(A) \cup \sigma(\beta)(A) \\
\sigma(\alpha^*)(A) &\doteq \bigcup_{k \in \mathbf{N}} \sigma(\alpha^k)(A)
\end{aligned}$$

and

$$\pi(\alpha)(A) = (-\sigma(\alpha)-)\,(A)$$

for $A \in \mathcal{P}(X)$. For an arbitrary topology $\mathcal{T}$ on X, we have:

$$\sigma(\alpha)\,(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}\,(\sigma(\alpha)(A)) \quad \text{for all subsets } A \subseteq X$$

and when $\mathcal{T}$ is a D-topology on X,

$$\pi(\alpha)\,(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}\,(\pi(\alpha)(A)) \quad \text{for all subsets } A \subseteq X$$

So the *continuity* schemes

$$\langle\alpha\rangle\mathbf{Cont}: \; \langle\alpha\rangle\Box\varphi \to \Box\langle\alpha\rangle\varphi \quad \text{and} \quad [\alpha]\mathbf{Cont}: \; [\alpha]\Box\varphi \to \Box[\alpha]\varphi$$

are true in all topological structures, and all D-topological structures, respectively[23]. Continuous analogs of the Hoare composition rules:

$$\frac{\psi \to \langle\alpha\rangle\Box\chi \quad \chi \to \langle\beta\rangle\Box\varphi}{\psi \to \langle\alpha\beta\rangle\Box\varphi} \quad \text{and} \quad \frac{\psi \to [\alpha]\Box\chi \quad \chi \to [\beta]\Box\varphi}{\psi \to [\alpha\beta]\Box\varphi}$$

are truth-preserving in all topological structures, and all D-topological structures, respectively. These later rules will be useful for an internal control automaton in looking for an action α such that:

$$\psi \to [\alpha]\Box\varphi$$

is entailed by the knowledge base of an agent, where ψ represents the current state and φ a goal.

The richer language of **TPDL** permits the expression of other interesting properties of actions. For example, the formula[24] $\varphi \to [\alpha^*]\varphi$ is true in a structure exactly

[23]For a D-topology $\mathcal{T}$, arbitrary intersections of open sets are open; this is what is needed for the continuity of the $[\alpha^*]$ operator.

[24]The converse $[\alpha^*]\varphi \to \varphi$ is **TPDL** provable.

when the set of φ states is closed under every iteration of every application of α; equivalently, $\|\varphi\|$ is the least fixed point of the operator $\pi(\alpha)$. Properties of this form are used in discussions of the viability of hybrid systems [KNRY95].

In addition to the Hilbert-style axiomatizations[25], I present tableaux proof systems for each of the logics. The orthodox treatment of tableaux, stemming from Smullyan's [Smu68], labels the nodes of a finitely-branching tree (or more generally, a directed graph) with sets of subformulas[26] of the formula at the root node. Such tableaux are developed according to rules capturing the semantics of the logic's connectives and operators, and development stops at a node when its label set contains a contradiction. In systems where tableaux are trees, there is usually a simple translation into a Gentzen-style sequent calculus: given a finite tableaux proof (all paths end in contradiction), turn it upside-down and "massage" a little to get a sequent calculus derivation, with the contradictions corresponding to axiom sequents. Orthodox tableaux have been used extensively in various modal, temporal and dynamic logics (see, for example, [Pra80], [Fi83], [BMP83], [BS84], [Wo85], [ESr88]) and have given rise to automata-based decision procedures for a number of logics (e.g. [Em85], [VW86]).

The tableaux systems presented here are in a different tradition. The system is an extension of the treatment of modal tableaux in [NS93] and [Ne90], which is in turn a descendant of the modal *prefixed tableaux* systems of Fitting [Fi72] and [Fi83] Ch. 8. The essential idea, which traces back to Fitch, is to add to the *formal language of proofs* symbols intended to name possible *worlds* in Kripke models, taking to heart the central idea from Beth [Be59] that the construction of a tableaux proof is an attempt to build a countermodel. So to give symbolic representation to such models, I include in the formal language of proofs not only symbols for possible worlds, but also symbols for both the accessibility *relation* and the *function*. A tableaux is a labelled binary tree where the labels are either *signed forcing assertions*:

$$T[\, t \Vdash \varphi \,] \quad \text{or} \quad F[\, t \Vdash \varphi \,]$$

or *accessibility assertions*:

$$t\mathbf{R}s$$

The t, s are functional terms generated from a stock of primitive world symbols $\mathbf{w}_i$, $i \in \mathbb{N}$. For the logics **S4F** and **S4C**, terms come from iterates of a single unary function

[25] For thoroughness, completeness proofs for each of the Hilbert-style proof systems are given. They are just variants of the standard constructions of Kripke models of maximal-consistent sets (with the usual extra work caused by the * operator).

[26] In the case of temporal and dynamic logics, subformulas are replaced by formulas in the Fischer-Ladner closure.

symbol $\mathbf{F}$, and have the form $\mathbf{F}^k(\mathbf{w}_i)$ for some $k \in \mathbb{N}$, while for the dynamic logic **TPDL**, the terms are of the form $(\mathbf{F}_{j_n} \circ \cdots \circ \mathbf{F}_{j_1})(\mathbf{w}_i)$, generated from compositions of unary functions symbols $\mathbf{F}_j$, $j \in \mathbb{N}$.

The tableaux development rules for signed forcing assertions break down (analyze) complex formulas φ and simultaneously build up (synthesize) complex terms t; these rules capture the various clauses of the definition of forcing for Kripke frames. For accessibility assertions, the development rules capture the reflexivity and transitivity of $\mathbf{R}$, and for the classes of **S4C** or **TPDL** tableaux, there is the continuity rule capturing the monotonicity of $\mathbf{F}$ or the $\mathbf{F}_j$ with respect to $\mathbf{R}$.

A tableaux with root $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$ is a *proof* of φ exactly when every path (branch) through the tableaux contains contradictory signed forcing assertions. A "failed" proof has a non-contradictory path $\mathtt{P}$ which naturally defines a *term frame* $\mathcal{K}_\mathtt{P}$ and a valuation $\eta_\mathtt{P}$ such that all the *assertions* on the path are true in the model $(\mathcal{K}_\mathtt{P}, \eta_\mathtt{P})$. So if the root entry is $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$, then φ is falsified at $\mathbf{w}_0$ in the model, while φ is satisfied at $\mathbf{w}_0$ in the model if the root entry is $T[\ \mathbf{w}_0 \Vdash \varphi\ ]$.

The domain of the term frame $\mathcal{K}_\mathtt{P}$ is generated from on the primitive world symbols $\mathbf{w}_i$ appearing in signed forcing assertions on $\mathtt{P}$ by closing under $\mathbf{F}$ or the $\mathbf{F}_j$; this ensures that the term constructor functions $t \mapsto \mathbf{F}(t)$ or $t \mapsto \mathbf{F}_j(t)$ are total, since these interpret the single action a, or the atomic actions $\Sigma = \{a_j \mid j \in \mathbb{N}\}$ in the language of **TPDL**. The relation $R_\mathtt{P}$ on $W_\mathtt{P}$ is the reflexive and transitive closure of the relation defined by the accessibility assertions occurring on the path $\mathtt{P}$; for **S4C** (**TPDL**) tableaux, we also take the $\mathbf{F}$-functional ($\mathbf{F}_j$-functional) closure of this relation, so that $(t, s) \in R_\mathtt{P}$ implies $(\mathbf{F}(t), \mathbf{F}(s)) \in R_\mathtt{P}$ $((\mathbf{F}_j(t), \mathbf{F}_j(s)) \in R_\mathtt{P})$. The tableaux development rules, specially that for $F[\ t \Vdash \Box\varphi\ ]$ assertions which force the introduction of a *new* primitive world symbol $\mathbf{w}_i$ and an entry $t\mathbf{R}\mathbf{w}_i$, ensure that the relation defined by the accessibility assertions occurring on a path $\mathtt{P}$ is always a *partial order*. Hence the closure $R_\mathtt{P}$ is always a partial order, and so the induced cone topology on $W_\mathtt{P}$ is a T_0 D-topology.

In proving completeness, we give a deterministic algorithm for constructing the *complete systematic tableaux* (CST) with root entry $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$, that applies every tableaux development rule that can be applied. The construction either terminates with a contradiction on every path, thus yielding a tableaux proof, or else continues indefinitely, producing an infinite tableaux. A non-contradictory path $\mathtt{P}$ through a CST naturally defines a valuation $\eta_\mathtt{P}$ for the term frame $\mathcal{K}_\mathtt{P}$ such that the formula φ is falsified at $\mathbf{w}_0$ in the model $(\mathcal{K}_\mathtt{P}, \eta_\mathtt{P})$.

To prove the finite model property for each of the logics, I define a quotient of the term frame $\mathcal{K}_\mathtt{P}$ for a non-contradictory path $\mathtt{P}$ through a CST. The quotient uses the set $S_\mathtt{P}(t)$ of signed forcing assertions on $\mathtt{P}$ with subject t, and identifies terms $t, s \in W_\mathtt{P}$ such that $S_\mathtt{P}(t) = S_\mathtt{P}(s)$. For the logics **S4F** and **S4C**, the sets $S_\mathtt{P}(t)$ are consistent subsets of the set of signed subformulas of the formula φ in the root entry, while for

TPDL, each $S_P(t)$ is a consistent subset of the signed Fischer-Ladner closure of φ. In either case, the quotient is finite, and the path valuation η_P faithfully passes through to the quotient. The sets of formulas $S_P(t)$ are essentially signed *Hintikka sets*, as used in the orthodox treatment of tableaux for modal logics; see, for example, [BS84], [BMP83].

In summary, each of the logics **S4F**, **S4C** and **TPDL** are complete for the class of their appropriate topological structures based on *countable state spaces with* T_0 *D-topologies*; they are also complete for the class of their appropriate topological structures based on *finite state spaces* (whose topologies are necessarily D-topologies). However, they cannot be complete for the intersection of the corresponding pair of classes since from [Kri63], §5.1, **S4** is *not* complete for the class of finite spaces with T_0 topologies, or equivalently, finite partially-ordered Kripke frames.

Just as this document was being completed, I obtained three short abstracts of work on "Dynamic Topological Logics" by Kremer, Mints and Rybakov [KrMi97], [Kre97], [KrMiR97]. They have independently developed **S4**-based dynamic logics, called **DTL**'s. Their logics include a "next" operator corresponding to the $[a]$ modality, for a single atomic action a, and a "star" operator corresponding to $[a^*]$ for atomic a. The abstracts announce axiomatizations of various fragments; for example, the star-free fragment of the logic $\mathbf{DTL}_{\mathcal{H}}$ of *homeomorphic* functions.

1.5 Formal Methods in Hybrid Systems

Much of the work in formal methods for hybrid systems focuses on various classes of automata. One of the foundational papers is [ACHH93], which introduces the class of *hybrid automata*. These are discrete transition systems on a finite set of control locations, with the behavior of real-valued variables in each location governed by differential equations and subject to an invariant condition. (See also [ACH+95], [He96].) Hybrid automata generalize a class of *timed automata* ([AD90], [AD94]) in which clock variables take real values. Real-time temporal logics have also been proposed as specification languages for hybrid systems, such as **TCTL** which extends the branching time temporal logic **CTL** by the addition of "clocks", with the semantics of the logic given by timed automata ([ACD93], [HeK97]).

This work proceeds on a somewhat different line of inquiry, since it takes as its starting point the idea that in a logic for hybrid systems, *topology* is an essential ingredient. Some common ground can be found in recent work by Henzinger and his coworkers [GHeJ97] on *robust timed automata*. That paper starts from the idea that an automaton model which represents an event occurring at an exact real time $t \in \mathbb{R}^+$

is not physically realizable; in any physical realization, the most that can be guaranteed is that the event occurs in an interval $(t-\epsilon, t+\epsilon)$. The acceptance and rejection conditions of timed automata are modified so that if a robust timed automaton accepts a trajectory, it must also accept every trajectory in an ϵ-neighborhood of that trajectory, and likewise for rejection. The underlying topology is a metric topology on the set of finite words $(\Sigma \times \mathbb{R}^+)^*$ (trajectories), where Σ is a finite alphabet.

1.6 Boundaries of this Investigation

Although drawing its motivation from hybrid control systems, and their "continuous dynamics", this work is primarily a formal study in modal logic. There are many points of interest not addressed; I envisage it as the beginning of a larger research project. In telegraphic form, topics of further investigation include the following:

- *A decent concrete example*, to illustrate how the logics can be practically used for hybrid control systems;

- *Getting more value out of the tableaux proof system*, including working out an explicit tableaux-based decision procedure which yields finite term models;

- *Extensions:* Look for decidable fragments of *first order* extensions, as well as *propositional* extensions such as poly-**S4** based logics of multiple topologies, say $\Box_{in}$ and $\Box_{out}$, in which case the continuity of $f^a : (X, \mathcal{T}_{in} \rightarrow (X, \mathcal{T}_{out})$ is captured by the scheme:

  $$[a]\Box_{out}\varphi \rightarrow \Box_{in}[a]\varphi$$

  or enriching the modalities of **TPDL** to represent the actions of multiple control agents, necessitating a treatment of concurrency;

- *Dealing with time as a distinguished coordinate*, since there are good reasons for wanting purely temporal modalities, drawing on the abstract treatment of time domains of Nicollin and Sifakis in [NiSi92] and [NiSiY92];

- *Topological completeness*: look for "real" topological completeness results like those of [MT44] and [RS63] for **S4** for dense-in-themselves metric spaces, starting by studying the finite subtopologies of such spaces;

- *Exploring the algebraic richness of the* **TPDL** *semantic structures*, including extending the work of Pratt [Pra79] and Kozen [Koz82] on dynamic algebras to topological dynamic algebras.

Chapter 2

The Logic S4F

2.1 Syntax and Topological Semantics

Definition 2.1.1 *Let $\mathcal{L}_{\Box a}$ be the propositional language generated from a countable set AP of atomic propositions, the propositional connectives $\neg$ (negation) and $\to$ (implication), and the modal operators $\Box$ and $[a]$.*

Within the language $\mathcal{L}_{\Box a}$, we can define in the usual way the propositional constants and the other classical propositional connectives in terms of $\neg$ and $\to$, the diamond operators $\Diamond$ and $\langle a\rangle$ as the classical duals of $\Box$ and $[a]$, respectively:

$$\begin{aligned}
\bot &\doteq \neg(p \to p) \qquad \text{for some } p \in AP\\
\top &\doteq \neg\bot\\
\varphi \wedge \psi &\doteq \neg(\varphi \to \neg\psi)\\[1ex]
\varphi \vee \psi &\doteq \neg\varphi \to \psi\\
\varphi \leftrightarrow \psi &\doteq (\varphi \to \psi) \wedge (\psi \to \varphi)\\
\Diamond\varphi &\doteq \neg\Box\neg\varphi\\
\langle a\rangle\varphi &\doteq \neg[a]\neg\varphi
\end{aligned}$$

Definition 2.1.2 *A* topological structure *for the propositional language $\mathcal{L}_{\Box a}$ is a triple $\mathfrak{T} = (X, \mathcal{T}, f)$ where*

- *$X \neq \emptyset$ is the state space;*
- *$\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on X (i.e. $\emptyset, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary unions and finite intersections); and*
- *$f : X \to X$ is a total function.*

Sets $U \in \mathcal{T}$ are called *open* (in $\mathcal{T}$), and $A \subseteq X$ is called *closed* (in $\mathcal{T}$) if $A = -U$ for some $U \in \mathcal{T}$. Note that at this stage, f is not assumed to be anything other than total. Our task, after all, is to discern the meaning of f being continuous with respect to $\mathcal{T}$.

Definition 2.1.3 *A* valuation *for a topological structure* $\mathfrak{T} = (X, \mathcal{T}, f)$ *is any map* $\xi : AP \to \mathcal{P}(X)$ *assigning a subset* $\xi(p) \subseteq X$ *to each* $p \in AP$. *Each such valuation uniquely extends to a valuation map* $\|\cdot\|_\xi : \mathcal{L}_{\Box a} \to \mathcal{P}(X)$, *satisfying the following clauses:*

$$\begin{aligned} \|p\|_\xi &= \xi(p) \\ \|\neg\varphi\|_\xi &= -\|\varphi\|_\xi \\ \|\varphi \to \psi\|_\xi &= -\|\varphi\|_\xi \cup \|\psi\|_\xi \\ \|\Box\varphi\|_\xi &= int_\mathcal{T}\left(\|\varphi\|_\xi\right) \\ \|[a]\varphi\|_\xi &= f^{-1}\left(\|\varphi\|_\xi\right) \end{aligned}$$

where $int_\mathcal{T}$ *is the* interior operator *determined by the topology* $\mathcal{T}$, *i.e. for all* $A \subseteq X$,

$$int_\mathcal{T}(A) = \bigcup\{U \in \mathcal{T} \mid U \subseteq A\}$$

and f^{-1} *is the* inverse-image operator *determined by the total function* f*:*

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}$$

Continuing the metaphor of a basis for a topology as a collections of lenses, one can think of the interior $int_\mathcal{T}(A)$ as that part of the set A which can be *discerned* through the lenses of $\mathcal{T}$. If $\mathcal{B}_\mathcal{T} = \{B_i \mid i \in I\}$ is a basis for $\mathcal{T}$, then:

$$int_\mathcal{T}(A) = \bigcup\{B_i \in \mathcal{B}_\mathcal{T} \mid B_i \subseteq A\}$$

So $\Box\varphi$ may be read as "discernibly φ". Thinking of a topology $\mathcal{T}$ as a collection of meaningful regions of X, $\Box\varphi$ may be read as "meaningfully φ".

Definition 2.1.4 *A* topological model *for* $\mathcal{L}_{\Box a}$ *is a pair* $(\mathfrak{T}, \xi)$, *where* $\mathfrak{T} = (X, \mathcal{T}, f)$ *is a topological structure for* $\mathcal{L}_{\Box a}$ *and* $\xi : AP \to \mathcal{P}(X)$ *is a valuation for* $\mathfrak{T}$.

Definition 2.1.5 *Let* $\varphi \in \mathcal{L}_{\Box a}$ *be a propositional formula.*

- φ *is* satisfied *at a state* $x \in X$ *in a topological model* $(\mathfrak{T}, \xi)$ *iff* $x \in \|\varphi\|_\xi$.
- φ *is* true *in a topological model* $(\mathfrak{T}, \xi)$, *written* $(\mathfrak{T}, \xi) \models \varphi$, *iff* $\|\varphi\|_\xi = X$;

- φ *is* valid *in a topological structure* $\mathfrak{T}$*, written* $\mathfrak{T} \models \varphi$*, iff for* all *valuations* ξ *for* $\mathfrak{T}$*, we have* $\|\varphi\|_\xi = X$*;*
- φ *is* topologically valid *iff* $\mathfrak{T} \models \varphi$ *for every topological structure* $\mathfrak{T} = (X, \mathcal{T}, f)$ *for* $\mathcal{L}_{\Box a}$.

The topological semantics for the defined constants, connectives and modal operators are as one would expect.

$$\begin{array}{rcl}
\|\bot\|_\xi &=& \emptyset \\
\|\top\|_\xi &=& X \\
\|\varphi \wedge \psi\|_\xi &=& \|\varphi\|_\xi \cap \|\psi\|_\xi \\
\|\varphi \vee \psi\|_\xi &=& \|\varphi\|_\xi \cup \|\psi\|_\xi \\
\|\Diamond\varphi\|_\xi &=& cl_\mathcal{T}\left(\|\varphi\|_\xi\right) \\
\|\langle a\rangle\varphi\|_\xi &=& (-f^{-1}-)\left(\|\varphi\|_\xi\right)
\end{array}$$

where $cl_\mathcal{T}$ is the closure operator determined by the topology $\mathcal{T}$, i.e. for any $A \subseteq X$,

$$\begin{array}{rcl}
cl_\mathcal{T}(A) &=& (-int_\mathcal{T}-)(A) \\
&=& \bigcap\{C \mid -C \in \mathcal{T} \text{and } A \subseteq C\}
\end{array}$$

Observe that for any topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ and valuation ξ for $\mathfrak{T}$,

$$\|\varphi \to \psi\|_\xi = X \quad \text{iff} \quad \|\varphi\|_\xi \subseteq \|\psi\|_\xi$$

More generally,

$$\|\varphi \to \psi\|_\xi = \{x \in X \mid \text{ if } \; x \in \|\varphi\|_\xi \; \text{ then } \; x \in \|\psi\|_\xi\,\}$$

The proposed reading of formulas of the form:

$$\psi \to [a]\varphi$$

as "action a always takes ψ states to φ states" is based on the fact that in any topological model $(\mathfrak{T}, \xi)$,

$$(\mathfrak{T}, \xi) \models \psi \to [a]\varphi \quad \text{iff} \quad \text{for all } x \in X, \text{ if } x \in \|\psi\|_\xi \text{ then } f(x) \in \|\varphi\|_\xi\,.$$

Note also that if I is a finite set and q_i, $i \in I$, are atomic propositions, the formulas:

$$\bigwedge_{i \in I}(q_i \leftrightarrow \Box q_i) \quad \text{and} \quad \bigvee_{i \in I} q_i$$

are true in a model $(\mathfrak{T}, \xi)$ exactly when $\{\|q_i\|_\xi\}_{i\in I}$ is a finite open cover of the topological space $(X, \mathcal{T})$.

Note that there are no restrictions on valuations $\xi : AP \to \mathcal{P}(X)$; i.e. each $\xi(p)$ is an *arbitrary* subset of X.

Definition 2.1.6 *Given a topological space* $(X, \mathcal{T})$, *let*

$$\mathfrak{B}_\mathcal{T}(X) = (\mathcal{P}(X), \cup, \cap, -, X, \emptyset, int_\mathcal{T})$$

denote the topological Boolean algebra [RS63] *with universe* $\mathcal{P}(X)$ *determined by* $(X, \mathcal{T})$*; i.e. the complete Boolean algebra of all subsets of* X*, equipped with the interior operator* $int_\mathcal{T} : \mathcal{P}(X) \to \mathcal{T} \subseteq \mathcal{P}(X)$*, which satisfies:*

$$\begin{array}{lrcl}
\text{(i)} & int_\mathcal{T}(A) & \subseteq & A \\
\text{(ii)} & int_\mathcal{T}(int_\mathcal{T}(A)) & = & int_\mathcal{T}(A) \\
\text{(iii)} & int_\mathcal{T}(A \cap B) & = & int_\mathcal{T}(A) \cap int_\mathcal{T}(B) \\
\text{(iv)} & int_\mathcal{T}(X) & = & X
\end{array}$$

In McKinsey and Tarski [MT44], topological Boolean algebras go by the name of *closure algebras*, and in the survey of Bull and Segerberg [BS84], and elsewhere, the term *modal algebra* is used for a broad class of algebras consisting of Boolean algebras equipped with unary operators satisfying various modal logic conditions. Moreover, from [MT44] and [RS63] III.4.3, every topological Boolean algebra $\mathcal{A} = (A, \vee, \wedge, -, 1, 0, \mathbf{I})$ is isomorphic with an algebra $\mathfrak{B}_\mathcal{T}(X)$ for some topological space $(X, \mathcal{T})$.

So algebraically, the topological models $(X, \mathcal{T}, f, \xi)$ correspond to evaluating formulas of $\mathcal{L}_{\Box a}$ in the topological Boolean algebra $\mathfrak{B}_\mathcal{T}(X)$, together with the inverse-image operator $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(X)$ of the total function f on X. As an operator on $\mathcal{P}(X)$, f^{-1} has particularly strong properties:

$$\begin{array}{rcl}
f^{-1}(-A) & = & -f^{-1}(A) \\
f^{-1}(A \cup B) & = & f^{-1}(A) \cup f^{-1}(B) \\
f^{-1}(A \cap B) & = & f^{-1}(A) \cap f^{-1}(B) \\
f^{-1}(\emptyset) & = & \emptyset \\
f^{-1}(X) & = & X
\end{array}$$

for all $A, B \in \mathcal{P}(X)$; i.e. f^{-1} preserves complements, unions, intersections, and the bottom ($\emptyset$) and top (X) elements. (The *totality* of f is required for the last equation since $f^{-1}(X) = dom(f)$.) Moreover, f^{-1} preserves *arbitrary* unions and intersections:

$$\begin{array}{rcl}
f^{-1}(\bigcup_{i\in I} A_i) & = & \bigcup_{i\in I} f^{-1}(A_i) \\
f^{-1}(\bigcap_{i\in I} A_i) & = & \bigcap_{i\in I} f^{-1}(A_i)
\end{array}$$

for any family of sets $\{A_i\}_{i\in I} \subseteq \mathcal{P}(X)$. Since f^{-1} preserves complements, it follows that

$$\|\langle a\rangle\varphi\|_\xi \;=\; f^{-1}\left(\|\varphi\|_\xi\right) \;=\; \|[a]\varphi\|_\xi$$

in all topological models $(\mathfrak{T}, \xi)$. When the apparatus of propositional dynamic logic **PDL** is adjoined in Chapter 5, we will have more to say about unary operators on the topological Boolean algebra $\mathfrak{B}_\mathcal{T}(X)$.

Recall that the standard Gödel embedding of Intuitionistic propositional logic **Int** into **S4** proceeds by "Boxing" all atomic propositions, i.e. $\Box p$, and defining Intuitionistic negation $\sim$ and Intuitionistic implication $\rightarrowtail$ by:

$$\begin{aligned} \sim\varphi &\triangleq \Box(\neg\varphi) \\ \varphi \rightarrowtail \psi &\triangleq \Box(\varphi \to \psi) \end{aligned}$$

So in an *Intuitionistic topological semantics* for the language $\mathcal{L}_{\Box a}$, we would require a restriction to a subclass of valuations $\xi : AP \to \mathcal{P}(X)$ such that $\xi(p) \in \mathcal{T}$ for all $p \in AP$, interpret negation as the interior of the complement, and interpret implication as the interior of classical implication. Interpreting $\Box$ by the interior operator would be vacuous, since all sets under consideration would already be open. To ensure that $f^{-1}(U)$ is open whenever U is open, we would need to *insist* that the function f be continuous. The advantage of taking **S4** rather than **Int** as our base logic is that in adding the $[a]$ modality, we can express *in the language* both the openness of sets and the continuity of a function.

2.2 Hilbert-style Proof System

Definition 2.2.1 *The Hilbert-style proof system for the logic* **S4F** *has the following axiom schemes, in the language* $\mathcal{L}_{\Box a}$*:*

CP :	*axioms of classical propositional logic in* $\mathcal{L}_{\Box a}$
$\Box$**K** :	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box$**T** :	$\Box\varphi \to \varphi$
$\Box$**4** :	$\Box\varphi \to \Box\Box\varphi$
$[a]$**K** :	$[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
$[a]$**F** :	$[a]\varphi \leftrightarrow \langle a\rangle\varphi$

where $\langle a\rangle\varphi$ *abbreviates* $\neg[a]\neg\varphi$, *together with the inference rules:*

$$\textit{modus ponens:} \quad \frac{\varphi,\ \varphi\to\psi}{\psi}$$

$$\Box\textit{-necessitation:} \quad \frac{\varphi}{\Box\varphi}$$

$$[a]\textit{-necessitation:} \quad \frac{\varphi}{[a]\varphi}$$

We write

$$\mathbf{S4F} \vdash_H \varphi$$

or say φ *is* $\mathbf{S4F}_H$ *provable, if the formula* $\varphi \in \mathcal{L}_{\Box a}$ *has an* **S4F** *Hilbert-style derivation.*

The axiom schemes $\Box\mathbf{K}$, $\Box\mathbf{T}$ and $\Box\mathbf{4}$, together with **CP**, and the rules of modus ponens and $\Box$-necessitation, constitute the standard Hilbert-style proof system for propositional **S4**. The $[a]\mathbf{F}$ axiom is of course equivalent to the conjunction of the schemes

$$[a]\mathbf{D}: \quad [a]\varphi \to \langle a\rangle\varphi$$

and

$$[a]\mathbf{D}_c: \quad \langle a\rangle\varphi \to [a]\varphi$$

The first is the well-known *deontic* scheme ("ought implies can"), while the second goes by the name of *determinism* in the dynamic logic literature. In virtue of the axiom $[a]\mathbf{K}$ and the rule of $[a]$-necessitation, $[a]$ is a *normal* modal (necessity) operator.

The following are $\mathbf{S4F}_H$ provable, for any formulas $\varphi, \psi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$, where if $k > 0$, $[a]^k\varphi$ denotes the formula $[a][a]...[a]\varphi$, with k iterations of the $[a]$ operator and if $k = 0$, then $[a]^k\varphi$ is just φ.

$$\begin{array}{ll}
[a]^k\neg: & \neg[a]^k\varphi \leftrightarrow [a]^k\neg\varphi \\
[a]^k\to: & [a]^k(\varphi\to\psi) \leftrightarrow ([a]^k\varphi \to [a]^k\psi) \\
[a]^k\wedge: & [a]^k(\varphi\wedge\psi) \leftrightarrow ([a]^k\varphi \wedge [a]^k\psi) \\
[a]^k\vee: & [a]^k(\varphi\vee\psi) \leftrightarrow ([a]^k\varphi \vee [a]^k\psi) \\
[a]^k\top: & [a]^k\top \\
[a]^k\bot: & [a]^k\bot \leftrightarrow \bot \\
[a]^k\Box: & [a]^k\Box\varphi \to [a]^k\varphi \\
[a]^k\Diamond: & [a]^k\varphi \to [a]^k\Diamond\varphi
\end{array}$$

$$
\begin{array}{ll}
\Box\Box: & \Box\Box\varphi \leftrightarrow \Box\varphi \\
\Box\wedge: & \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi) \\
\Box\vee: & (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi) \\
\Box\top: & \Box\top \\
\Diamond\mathbf{T}: & \varphi \rightarrow \Diamond\varphi \\
\Diamond\Diamond: & \Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi \\
\Diamond\wedge: & \Diamond(\varphi \wedge \psi) \rightarrow (\Diamond\varphi \wedge \Diamond\psi) \\
\Diamond\vee: & \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi) \\
\Diamond\bot: & \Diamond\bot \leftrightarrow \bot
\end{array}
$$

The following are admissible inference rules in $\mathbf{S4F}_H$, for any formulas $\varphi, \psi, \chi \in \mathcal{L}_{\Box a}$ and $k, l \in \mathbb{N}$:

$$
\begin{array}{ll}
[a]^k\textit{-necessitation:} & \dfrac{\varphi}{[a]^k\varphi} \\[2ex]
\textit{Monotonicity of } [a]^k\textit{:} & \dfrac{\varphi \rightarrow \psi}{[a]^k\varphi \rightarrow [a]^k\psi} \\[2ex]
\textit{Hoare composition:} & \dfrac{\varphi \rightarrow [a]^k\chi,\ \chi \rightarrow [a]^l\psi}{\varphi \rightarrow [a]^{k+l}\psi}
\end{array}
$$

Observe that there are no axioms for **S4F** containing both $\Box$ and $[a]$, so the behaviors of the two modalities are quite independent and the logic can be thought of as a "direct product" of **S4** and **F**. When we adjoin a true bimodal axiom such as

$$\textbf{Cont}: \quad [a]\Box\varphi \rightarrow \Box[a]\varphi$$

the result is a richer "amalgamated product" of **S4** and **F**.

Proposition 2.2.2 Topological Soundness of **S4F** Hilbert-style proof system
For all formulas φ of $\mathcal{L}_{\Box a}$, if $\mathbf{S4F} \vdash_H \varphi$ *then φ is topologically valid.*

Proof. The topological validity of the **S4** axioms for $\Box$ plus the validity-preservation of modus ponens $\Box$-necessitation follow trivially from the properties of the interior operator; see [McK41], [MT44]. The semantical validity of the $[a]$-necessitation rule translates as

$$\|\varphi\|_\xi = X \;\text{ implies }\; f^{-1}\left(\|\varphi\|_\xi\right) = X$$

and the equation $f^{-1}(X) = X$ holds exactly when $f : X \rightarrow X$ is a *total* function. The validity of the **F** axioms for $[a]$ are immediate from the properties of the inverse-image operator. ■

2.3 Kripke Semantics

Although Tarski and McKinsey's topological and algebraic semantics for **S4** predate Kripke's relational semantics[1], the interpretation of modal operators via binary "accessibility" relations is now the standard approach. We now define Kripke models for the language $\mathcal{L}_{\Box a}$, and in the following section, we investigate transformations between the two types of models.

Definition 2.3.1 *A* Kripke frame *for $\mathcal{L}_{\Box a}$ is a triple $\mathcal{K} = (W, R, F)$, where*

- *$W \neq \emptyset$ is a set of "worlds";*
- *$R \subseteq W \times W$ is a reflexive and transitive binary relation on W; and*
- *$F : W \to W$ is a total function on W.*

A Kripke frame $\mathcal{K} = (W, R, F)$ is called finite *iff W is a finite set.*

By standard arguments, reflexive and transitive binary relations capture precisely the **S4** $\Box$ modality. As in [Lem77], §4, pp.60-61, a total function $F : W \to W$ is used to interpret the $[a]$ modality. If one prefers to interpret modalities with a binary relation on W, take $Q = graph(F)$. Then as a binary relation, Q is both "total" and "functional", i.e. for all $w \in W$, there *exists a unique* $v \in W$ such that $(w, v) \in Q$. The "totality" or "serial" condition: every $w \in W$ has *at least* one Q-successor, is characteristic for the deontic scheme:

$$[a]\mathbf{D}: \quad [a]\varphi \to \langle a\rangle\varphi$$

The converse scheme:

$$[a]\mathbf{D}_c: \quad \langle a\rangle\varphi \to [a]\varphi$$

is characterized by the "functionality" or "determinism" condition: every $w \in W$ has *at most* one Q-successor.

[1]Relational semantics for **S4** can also be extracted as a special case from Theorem's 3.3 and 3.5 of Jónsson and Tarski's work on Boolean algebras with operators [JT51], although as noted in [Kri63], Kripke's semantics were developed independently of these results.

Definition 2.3.2 *A* valuation *for a Kripke frame $\mathcal{K} = (W, R, F)$ is a map $\eta : W \to \mathcal{P}(AP)$ assigning a set of atomic propositions $\eta(w) \subseteq AP$ to each world $w \in W$. Each such valuation for $\mathcal{K}$ determines a forcing relation $\Vdash^{\mathcal{K}}_{\eta} = \Vdash_{\eta} \subseteq W \times AP$ defined by*

$$w \Vdash_{\eta} p \quad \textit{iff} \quad p \in \eta(w)$$

which uniquely extends a forcing relation $\Vdash_{\eta} \subseteq W \times \mathcal{L}_{\Box a}$ (with the same name) on all formulas of $\mathcal{L}_{\Box a}$, by the following clauses:

(i) $w \Vdash_{\eta} \neg\varphi$ *iff* $w \nVdash_{\eta} \varphi$;

(ii) $w \Vdash_{\eta} \varphi \to \psi$ *iff* $w \nVdash_{\eta} \varphi$ *or* $w \Vdash_{\eta} \psi$;

(iii) $w \Vdash_{\eta} \Box\varphi$ *iff for all* $v \in W$, *if* $(w, v) \in R$ *then* $v \Vdash_{\eta} \varphi$;

(iv) $w \Vdash_{\eta} [a]\varphi$ *iff* $F(w) \Vdash_{\eta} \varphi$.

for all $w \in W$, and all $\varphi, \psi \in \mathcal{L}_{\Box a}$.

If $Q = graph(F)$, then by the total functionality of Q, this last clause is equivalent to

$$w \Vdash_{\eta} [a]\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w, v) \in Q \text{ then } v \Vdash_{\eta} \varphi.$$

Definition 2.3.3 *A* Kripke model *for $\mathcal{L}_{\Box a}$ is a pair $(\mathcal{K}, \eta)$, where $\mathcal{K} = (W, R, F)$ is a frame for $\mathcal{L}_{\Box a}$ and $\eta : W \to \mathcal{P}(AP)$ is a valuation for $\mathcal{K}$.*

Definition 2.3.4 *Let φ be a propositional formula of $\mathcal{L}_{\Box a}$.*

- *φ is* satisfied *(or* forced*) at a world $w \in W$ in a Kripke model $(\mathcal{K}, \eta)$ iff $w \Vdash^{\mathcal{K}}_{\eta} \varphi$;*
- *φ is* true *in a Kripke model $(\mathcal{K}, \eta)$, written $(\mathcal{K}, \eta) \Vdash \varphi$, iff for* all *worlds $w \in W$, we have $w \Vdash^{\mathcal{K}}_{\eta} \varphi$;*
- *φ is* valid *in a frame $\mathcal{K}$, written $\mathcal{K} \Vdash \varphi$, iff for* all *valuations $\eta : W \to \mathcal{P}(AP)$ for $\mathcal{K}$, we have $(\mathcal{K}, \eta) \Vdash \varphi$;*
- *φ is* Kripke valid *iff for all frames $\mathcal{K}$ for $\mathcal{L}_{\Box a}$, $\mathcal{K} \Vdash \varphi$.*

Proposition 2.3.5 Kripke Soundness of **S4F** Hilbert-style proof system
For all formulas φ of $\mathcal{L}_{\Box a}$, if **S4F** $\vdash_H \varphi$ *then φ is Kripke valid.*

Proof. The required verification is that each of the axioms of $\mathbf{S4F}_H$ are Kripke valid, and that the inference rules of $\mathbf{S4F}_H$ preserve Kripke validity. For the axioms **CP** of classical propositional logic and for *modus ponens*, this is trivial. The verification for the **S4** axioms **K**, **T** and **4**, and the $\Box$-necessitation rule follow the standard proof of soundness of the class of transitive and reflexive frames for **S4**; see, for example, [HC96], pp.56-57. For the $[a]$-necessitation rule, suppose φ is Kripke valid, let $\mathcal{K} = (W, R, F)$ be a frame for $\mathcal{L}_{\Box a}$, and let η be a valuation for $\mathcal{K}$. Since φ is Kripke valid and $F(w) \in W$ since F is total, we have $F(w) \Vdash_\eta \varphi$. Hence $w \Vdash_\eta \varphi$. Hence $[a]\varphi$ is also Kripke valid. The verification of the validity of the $[a]\mathbf{K}$ and $[a]\mathbf{F}$ axioms is also straightforward, taking as a starting point the fact that for any formula φ and any $w \in W$, either $F(w) \Vdash_\eta \varphi$ or $F(w) \nVdash_\eta \varphi$, and then crunching through the definitions of forcing for $\neg$, $\rightarrow$ and $[a]$. ■

The Hilbert-style proof system for **S4F** is "obviously" complete with respect to the Kripke semantics. For the purposes of the "completeness" of this investigation, we give the standard but "cheap" maximal-consistent sets proof, using as a template the generic treatment of modal logics in [Gol92], Part One, §3. Building a Kripke model out of maximal-consistent sets of formulas doesn't take much work, but then the resulting structure doesn't have much in the way of intuitive content. Our real interest is in the tableaux proof system developed in Chapter 4, where the proof of completeness takes rather more effort, but the reward is a more intuitive and conceptually transparent Kripke model constructed out of functional terms appearing on a path through the tableaux.

Definition 2.3.6 *Let $\mathcal{L}$ be a propositional language generated from a countable set of atomic propositions. A set of formulas $\Lambda \subseteq \mathcal{L}$ is called a* logic *(in $\mathcal{L}$) iff every tautology in $\mathcal{L}$ is in Λ, and Λ is closed under modus ponens. A formula $\varphi \in \mathcal{L}$ is a* theorem *of Λ, written $\vdash_\Lambda \varphi$, exactly when $\varphi \in \Lambda$.*

Let U be any set of formulas of $\mathcal{L}$.

A formula $\varphi \in \mathcal{L}$ is Λ-deducible *from U, written $U \vdash_\Lambda \varphi$, iff there is a finite number of formulas $\psi_1, ..., \psi_n \in U$ such that: $\vdash_\Lambda (\psi_1 \wedge ... \wedge \psi_n) \rightarrow \varphi$.*

U is called Λ-consistent *iff there is some formula of $\mathcal{L}$ that is* not *Λ-deducible from U; equivalently, $U \nvdash_\Lambda \perp$.*

U is called maximal *Λ-consistent iff U is Λ-consistent and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in U$ or $\neg\varphi \in U$.*

A formula $\varphi \in \mathcal{L}$ is called Λ-consistent *iff the set $\{\varphi\}$ is Λ-consistent; equivalently, $\nvdash_\Lambda \neg\varphi$.*

We let **S4F** (and likewise for subsequent extensions) denote the set of all formulas $\varphi \in \mathcal{L}_{\Box a}$ such that $\mathbf{S4F} \vdash_H \varphi$. In particular, every maximal **S4F**-consistent set $U \subseteq \mathcal{L}_{\Box a}$ contains all instances of the axiom schema of **S4F**.

By Lindenbaum's Lemma, every Λ-consistent set of formulas can be extended to a maximal Λ-consistent set. It follows that

$$\Lambda = \bigcap \{ U \subseteq \mathcal{L} \mid U \text{ is maximal } \Lambda\text{-consistent } \}.$$

Proposition 2.3.7 Kripke Completeness of **S4F** Hilbert-style proof system
There exists a Kripke model $(\mathcal{K}_0, \eta_0)$ *such that for all formulas* φ *of* $\mathcal{L}_{\Box a}$,

$$(\mathcal{K}_0, \eta_0) \Vdash \varphi \quad \textit{iff} \quad \mathbf{S4F} \vdash_H \varphi$$

Proof. The canonical Kripke frame $\mathcal{K}_0 = (W_0, R_0, F_0)$ is defined as follows:

$$\begin{array}{c} W_0 \doteq \{ U \subseteq \mathcal{L}_{\Box a} \mid U \text{ is maximal } \mathbf{S4F}\text{-consistent} \} \\ (U, V) \in R_0 \ \text{ iff } \ (\forall \varphi \in \mathcal{L}_{\Box a})[\, \Box\varphi \in U \ \Rightarrow \ \varphi \in V \,] \\ F_0(U) = V \ \text{ iff } \ (\forall \varphi \in \mathcal{L}_{\Box a})[\, [a]\varphi \in U \ \Rightarrow \ \varphi \in V \,] \end{array}$$

and the canonical valuation $\eta_0 : W_0 \to \mathcal{P}(AP)$ for $\mathcal{K}_0$ is defined by:

$$p \in \eta_0(U) \quad \text{iff} \quad p \in U$$

for all $p \in AP$ and $U \in W_0$. By standard arguments, the $\Box$**T** axiom ensures that R_0 is reflexive, and the $\Box$**4** axiom ensures that R_0 is transitive. To establish that F_0 is well-defined, it suffices to show for maximal **S4F**-consistent sets U, V,

$$\begin{array}{l} \text{if } \ (\forall \varphi \in \mathcal{L}_{\Box a})[\, [a]\varphi \in U \ \Rightarrow \ \varphi \in V \,] \\ \text{then } \ (\forall \varphi \in \mathcal{L}_{\Box a})[\, [a]\varphi \notin U \ \Rightarrow \ \varphi \notin V \,] \end{array}$$

in which case F_0 satisfies:

$$F_0(U) = \{ \varphi \in \mathcal{L}_{\Box a} \mid [a]\varphi \in U \}$$

and is thus a well-defined total function; equivalently, $F_0(U)$ is maximal **S4F**-consistent whenever U is. Assume the antecedent holds and $[a]\varphi \notin U$. Hence $\neg[a]\varphi \in U$. Since **S4F** $\vdash_H \neg[a]\varphi \leftrightarrow [a]\neg\varphi$, from the $[a]$**F** axiom, we must have $[a]\neg\varphi \in U$, hence by assumption, $\neg\varphi \in V$. Hence $\varphi \notin V$.

An easy induction on formulas establishes the "Truth Lemma": for all $\varphi \in \mathcal{L}_{\Box a}$ and sets $U \in W_0$,

$$U \Vdash_{\eta_0} \varphi \quad \text{iff} \quad \varphi \in U$$

Then since

$$\mathbf{S4F} \vdash_H \varphi \quad \text{iff} \quad \varphi \in \bigcap \{ U \subseteq \mathcal{L}_{\Box a} \mid U \text{ is maximal } \mathbf{S4F}\text{-consistent } \}$$

we have

$$(\mathcal{K}_0, \eta_0) \Vdash \varphi \ \text{ iff } \ \mathbf{S4F} \vdash_H \varphi$$

as required. ■

2.4 Duality of Topological and Kripke Semantics

In this section, we investigate the transformation of a topological structure into a Kripke frame, and conversely. In the process, we rediscover a correspondence that can be derived from Grzegorczyk's [Grz67]: the subclass of topological structures with D-topologies are the natural duals of Kripke frames for $\mathcal{L}_{\Box a}$; moreover, the duality transformation gives a semantically faithful bijection between Kripke models and D-topological models. This duality transformation will establish the correspondence:

$$\begin{array}{rcl} \text{Kripke frame } \mathcal{K} = (W, R, F) & & \text{topological structure } \mathfrak{T} = (X, \mathcal{T}, f) \\ \text{worlds} & \approx & \text{points in state space} \\ \text{accessibility or "meaning" relation} & \approx & \text{topology} \\ \text{function} & \approx & \text{function} \\ \text{forcing} & \approx & \text{membership} \end{array}$$

We begin the construction of a reflexive and transitive relation on X from a topology $\mathcal{T}$ on X, in the form presented in Scott's [Sco72].

Definition 2.4.1 *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a topological structure for $\mathcal{L}_{\Box a}$. Define a binary relation $R_{\mathcal{T}} \subseteq X \times X$ by:*

$$(x, y) \in R_{\mathcal{T}} \text{ iff } (\forall U \in \mathcal{T})[\, x \in U \Rightarrow y \in U \,]$$

It is immediate that $R_{\mathcal{T}}$ is reflexive and transitive, and since $f : X \to X$ is total, $\mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$ is a Kripke frame for $\mathcal{L}_{\Box a}$. The frame $\mathcal{K}_{\mathfrak{T}}$ is called the Kripke frame induced by $\mathfrak{T}$, and $R_{\mathcal{T}}$ is called the relation induced by the topology $\mathcal{T}$.

Thinking of a topology $\mathcal{T}$ as a collection of *meaningful regions* of X, $(x, y) \in R_{\mathcal{T}}$ says that y has all the same (topological) meaning as x has; $R_{\mathcal{T}}$ is the *meaning relation* of $\mathcal{T}$. In the category-theoretic language of frames and locales, $R_{\mathcal{T}}$ is known as the *specialization pre-order* ([Joh82], §II.1.8, [Smy92], §4.1, and [Vi89], §7.1).

In [Grz67] (Lemma 1), a relation $R'_{\mathcal{T}}$ is defined by:

$$(x, y) \in R'_{\mathcal{T}} \quad \text{iff} \quad x \in cl_{\mathcal{T}}(\{y\})$$

Observe that if $(x, y) \notin R'_{\mathcal{T}}$ then $x \in -cl_{\mathcal{T}}(\{y\}) = int_{\mathcal{T}}(-\{y\})$, so $U = int_{\mathcal{T}}(-\{y\})$ is an open set containing x and not y, hence $(x, y) \notin R_{\mathcal{T}}$. Conversely, if $(x, y) \notin R_{\mathcal{T}}$, then there exists an open set $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$. Hence $A = -U$ is a closed set such that $x \notin A$ and $y \in A$. Since $cl_{\mathcal{T}}(\{y\})$ is the intersection of all closed sets containing y, we must have $x \notin cl_{\mathcal{T}}(\{y\})$, and so $(x, y) \notin R'_{\mathcal{T}}$. Hence the Grzegorczyk and Scott definitions are equivalent.

Recall the following definitions of separation properties for topological spaces.

Definition 2.4.2 *Let $(X, \mathcal{T})$ be a topological space.*

$(X, \mathcal{T})$ is T_0 iff for all $x, y \in X$ with $x \neq y$, there exists $U \in \mathcal{T}$ such that either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

$(X, \mathcal{T})$ is T_1 iff for all $x, y \in X$ with $x \neq y$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$.

$(X, \mathcal{T})$ is Hausdorff *or T_2 iff for all $x, y \in X$ with $x \neq y$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.*

Clearly, Hausdorff $\Rightarrow$ T_1 $\Rightarrow$ T_0. In most texts in topology or analysis or their applications, the Hausdorff property is taken as a bare minimum from Chapter 2 onwards; for example, all metric spaces are Hausdorff. Another way of viewing these properties is to see that in a topological space $(X, \mathcal{T})$ satisfying any separation property *stronger* than T_0, "meaning" or "information" is concentrated at *single points.*

Proposition 2.4.3 *Let $(X, \mathcal{T})$ be a topological space, and let $R_{\mathcal{T}}$ be the relation induced by $\mathcal{T}$.*

(i) *$(X, \mathcal{T})$ is T_1 iff $R_{\mathcal{T}}$ is the identity relation on X;*

(ii) *$(X, \mathcal{T})$ is T_0 iff $R_{\mathcal{T}}$ is a partial order on X.*

Proof. The equivalence (i) is immediate, since $(X, \mathcal{T})$ is T_1 iff for all $x, y \in X$ with $x \neq y$, $(x, y) \notin R_{\mathcal{T}}$. Equivalence (ii) is also trivial, since $(X, \mathcal{T})$ is T_0 iff for all $x, y \in X$ with $x \neq y$, either $(x, y) \notin R_{\mathcal{T}}$ or $(y, x) \notin R_{\mathcal{T}}$, so the T_0 property is equivalent to the anti-symmetry of $R_{\mathcal{T}}$ [2]. ■

Note that if $\mathcal{T}$ is the discrete topology on X, then $R_{\mathcal{T}} = \mathbf{1}_X$, the identity relation on X. So under the transformation $\mathfrak{T} = (X, \mathcal{T}, f) \mapsto \mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$, from an arbitrary topological structure to its induced Kripke frame, all structures with topologies with T_1 or stronger get collapsed with structures with the discrete topology. To get a one-one correspondence between Kripke frames and topological structures, we clearly need to focus on a smaller class of topologies.

Definition 2.4.4 *A topology $\mathcal{T}$ on a space X is a D-*topology *iff for each $x \in X$, the set:*

$$B_x = \bigcap \{U \in \mathcal{T} \mid x \in U\}$$

*is open in $\mathcal{T}$, in which case $(X, \mathcal{T})$ is called a D-*space.

[2] In [Sco72], Scott's interest was in partial orders, so the $R_{\mathcal{T}}$ relation was defined for T_0 spaces.

Proposition 2.4.5 *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent:*

(i) *$\mathcal{T}$ is a D-topology on X;*

(ii) *the intersection of any family of open sets is open;*

(iii) *the union of any family of closed sets is closed;*

(iv) *for all $A \subseteq X$, $\quad cl_{\mathcal{T}}(A) = \bigcup_{y \in A} cl_{\mathcal{T}}(\{y\})$;*

(v) *$(\mathcal{T}, \subseteq, \cup, \cap, X, \emptyset, \bigcup, \bigcap)$ is a complete lattice of sets.*

Proof. (ii) says that $\mathcal{T}$ is closed under arbitrary intersections (as well as arbitrary unions), so (ii) $\Leftrightarrow$ (v) is immediate. (ii) $\Leftrightarrow$ (iii) comes from taking complements. (iii) $\Rightarrow$ (iv) is immediate, and (iv) $\Rightarrow$ (iii) is an easy exercise. (ii) $\Rightarrow$ (i) is trivial, since B_x is the intersection of all open sets containing x. For the converse, suppose $\mathcal{T}$ is a D-topology on X, and let $\{V_i\}_{i \in I} \subseteq \mathcal{T}$ be any family of open sets. Suppose $x \in \bigcap_{i \in I} V_i$, and let $B_x = \bigcap\{U \in \mathcal{T} \mid x \in U\}$. Then since $x \in V_i$ for all $i \in I$, we must have $B_x \subseteq \bigcap_{i \in I} V_i$. Since $B_x \in \mathcal{T}$ and

$$int_{\mathcal{T}}\left(\bigcap\nolimits_{i \in I} V_i\right) = \bigcup\{U \in \mathcal{T} \mid U \subseteq \bigcap\nolimits_{i \in I} V_i\}$$

we have $x \in int_{\mathcal{T}}\left(\bigcap_{i \in I} V_i\right)$. Hence $\bigcap_{i \in I} V_i$ is open, as required. ■

In [Grz67], §1, a topological space $(X, \mathcal{T})$ is said to be "totally distributive" when condition (iv) above is satisfied. The following proposition summarizes the important properties of D-topologies.

Proposition 2.4.6 *If $(X, \mathcal{T})$ is a D-space, $B_x = \bigcap\{U \in \mathcal{T} \mid x \in U\}$ for each $x \in X$, and $R_{\mathcal{T}}$ is the relation induced by $\mathcal{T}$, then:*

(a) *The family $\{B_x\}_{x \in X}$ is a basis for the topology $\mathcal{T}$.*

(b) *For all $x, y \in X$, $\quad (x, y) \in R_{\mathcal{T}}$ iff $B_y \subseteq B_x$.*

(c) *For all $x \in X$, $\quad B_x = \{y \in X \mid (x, y) \in R_{\mathcal{T}}\}$*

(d) *For all $A \subseteq X$,*

$$int_{\mathcal{T}}(A) = \{x \in X \mid (\forall y \in X)[\text{ if } (x, y) \in R_{\mathcal{T}} \text{ then } y \in A\,]\}$$

and

$$cl_{\mathcal{T}}(A) = \{x \in X \mid (\exists y \in X)[\,(x, y) \in R_{\mathcal{T}} \text{ and } y \in A\,]\}$$

(e) *For all* $x \in X$ *and* $\{V_i\}_{i \in I} \subseteq \mathcal{T}$,

$$\textit{if } B_x \subseteq \bigcup_{i \in I} V_i, \textit{ then } B_x \subseteq V_i \textit{ for some } i \in I$$

hence B_x *is* fully join-irreducible *in* $\mathcal{T}$, *considered as a complete lattice of (open) sets. Moreover, if* U *is non-empty and fully join-irreducible in* $\mathcal{T}$, *then* $U = B_x$ *for some* $x \in X$.

(f) $(X, \mathcal{T})$ *is Hausdorff iff* $\mathcal{T}$ *is the discrete topology on* X.

(g) $(X, \mathcal{T})$ *is* T_1 *iff* $\mathcal{T}$ *is the discrete topology on* X.

(h) $(X, \mathcal{T})$ *is* T_0 *iff for all* $x, y \in X$, $B_x = B_y$ *implies* $x = y$.

Proof. For (a), suppose $U \in \mathcal{T}$ and $x \in U$. Then $B_x \subseteq U$ and $x \in B_x$. Hence $\{B_x\}_{x \in X}$ is a basis for $\mathcal{T}$. For (b), observe that $B_y \subseteq B_x$ iff every open set containing x also contains y. (c) follows immediately from (b). For (d), using Grzegorczyk's equivalent definitions of $R_{\mathcal{T}}$ and a D-topology, the closure equation follows immediately from:

$$cl_{\mathcal{T}}(A) \;=\; \{x \in X \mid (\exists y \in X)[\, x \in cl_{\mathcal{T}}(\{y\}) \text{ and } y \in A\,]\}$$

The interior equation comes by taking complements.

For (e), fix $x \in X$ and $\{V_i\}_{i \in I} \subseteq \mathcal{T}$, and suppose $B_x \subseteq \bigcup_{i \in I} V_i$. Then since the B_y's form a basis for $\mathcal{T}$, each V_i is a union of B_y's; indeed, $V_i = \bigcup\{B_y \mid B_y \subseteq V_i\}$. Hence

$$B_x \subseteq \bigcup\{B_y \mid B_y \subseteq V_i \text{ for some } i \in I\}.$$

Now since $x \in B_x$, we have $x \in B_y$ for some y and V_i such that $B_y \subseteq V_i$. Since $x \in B_y$ iff $B_x \subseteq B_y$, it follows that $B_x \subseteq B_y \subseteq V_i$ for some $i \in I$.

Recall from lattice theory that $U \in \mathcal{T}$ is *fully join-irreducible* in $\mathcal{T}$, considered as a complete lattice, if for all $\{V_i\}_{i \in I} \subseteq \mathcal{T}$,

$$\text{if } U = \bigcup_{i \in I} V_i, \text{ then } U = V_i \text{ for some } i \in I$$

It is readily shown (see, for example, Definition XII.4.3 of [BD74] and the discussion following it) that this last condition is equivalent to:

$$\text{if } U \subseteq \bigcup_{i \in I} V_i, \text{ then } U \subseteq V_i \text{ for some } i \in I$$

for all $\{V_i\}_{i\in I} \subseteq \mathcal{T}$.

For the converse, suppose $U \in \mathcal{T}$ is fully join-irreducible in $\mathcal{T}$. Then since $U = \bigcup\{B_x \mid x \in U\}$, we have $U = B_x$ for some $x \in U$, by join-irreducibility.

For (f), if $(X, \mathcal{T})$ is Hausdorff, then for any $x \in X$,

$$B_x = \bigcap\{U \in \mathcal{T} \mid x \in U\} = \{x\} = cl_{\mathcal{T}}(\{x\})$$

hence $\{x\}$ is open (and closed), so $\mathcal{T}$ is the discrete topology. Conversely, the discrete topology is trivially Hausdorff.

For (g), assume $(X, \mathcal{T})$ is T_1. Then given any $U \in \mathcal{T}$ with $x_1, x_2 \in U$ and $x_1 \neq x_2$, there exist sets $U_1, U_2 \in \mathcal{T}$ such that $x_i \in U_i$ for $i = 1, 2$, but $x_1 \notin U_2$ and $x_2 \notin U_1$. Take $V = U \cap U_1$ and $W = U \cap U_2$. Then $U = V \cup W$, but if $U = V$ then $U = U \cap U_1$, so $U \subseteq U_1$, hence $x_2 \notin U$, contradicting the assumption that $x_1, x_2 \in U$. Similarly, if $U = W$ then $U \subseteq U_2$ hence $x_1 \notin U$, again a contradiction. Hence U is not a join irreducible of the lattice $\mathcal{T}$. So no set $U \in \mathcal{T}$ containing more than one point can be a join-irreducible of (or fully join-irreducible in) the lattice $\mathcal{T}$. Since the B_x's are fully join-irreducible in $\mathcal{T}$, the only possibility is $B_x = \{x\}$ for each $x \in X$, in which case the topology $\mathcal{T}$ is discrete. Conversely, the discrete topology is trivially T_1.

Finally, for (h), we have from Proposition 2.4.3 that $(X, \mathcal{T})$ is T_0 iff $R_{\mathcal{T}}$ is a partial order on X. Then use part (b). ■

Now we need to go the other way: we get a topology $\mathcal{T}_R$ from a reflexive and transitive binary relation R.

Definition 2.4.7 *Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box a}$. Define $\mathcal{T}_R$ to be the topology on W which has as its basic open sets the collection of all sets:*

$$B_w = \{v \in W \mid (w, v) \in R\}$$

So B_w is the set of all v that are R-accessible from w. Since $F : W \to W$ is a total function, $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ is a topological structure for $\mathcal{L}_{\Box a}$. The structure $\mathfrak{T}_{\mathcal{K}}$ is called the topological structure induced by $\mathcal{K}$, and $\mathcal{T}_R$ is the topology on W induced by R.

Note that $w \in B_w$ (by the reflexivity of R), and $v \in B_w$ implies $B_v \subseteq B_w$ (by the transitivity of R); conversely, $B_v \subseteq B_w$ implies $v \in B_w$, since $v \in B_v$. To confirm that the collection $\mathcal{B} = \{B_w \mid w \in W\}$ is suitable as a *basis* for a topology (and not merely a sub-basis), observe that if both $u \in B_w$ and $u \in B_v$, then we have $u \in B_u \subseteq B_w \cap B_v$.

The topology $\mathcal{T}_R$ is a generalization, to reflexive and transitive relations, of what is called in [Joh82], §II.1.8, the *Alexandroff topology* $\mathcal{T}_{\leq}$ on a partially-ordered set $(P, \leq)$

generated by the "upper cones" with respect to $\leq$ (see also [Smy92], §2.4). $\mathcal{T}_R$ also goes by the name "cone topology" [Mi95]. In Alexandroff's *Combinatorial Topology* [Ale56], I, §6.3, the lower cone $C_p = \{q \in P \mid q \leq p\}$ is called the "combinatorial closure" of the point $p \in P$. It is readily verified that in the topological space $(W, \mathcal{T}_R)$, where R is reflexive and transitive, the lower cone C_w satisfies:

$$C_w \doteq \{v \in W \mid (v, w) \in R\} = cl_{\mathcal{T}_R}(\{w\})$$

Note that in the *Scott topology* $\mathcal{T}_S$ on a partially-ordered set $(P, \leq)$ ([Sco72], §2), a set $U \subseteq P$ is open iff:

(1.) for all $p, q \in P$, if $p \in U$ and $p \leq q$ then $q \in U$; and

(2.) for all directed sets $D \subseteq P$, if $\sqcup D$ exists and $\sqcup D \in U$ then $D \cap U \neq \emptyset$;

where $\sqcup D$ is the sup or least upper bound of D. Open sets in the Alexandroff topology on $(P, \leq)$ are characterized by condition (1.) only, so the Scott topology $\mathcal{T}_S$ is a *sub-topology* of the Alexandroff topology $\mathcal{T}_{\leq}$ on P; i.e. $\mathcal{T}_{\leq}$ has more open sets than $\mathcal{T}_S$. Scott topologies are only really appropriate if $(P, \leq)$ is a *dcpo*, in which case $\sqcup D$ exists for every directed set $D \subseteq P$.

The next proposition records the relevant properties of the topology $\mathcal{T}_R$.

Lemma 2.4.8 *Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box_a}$ and let $\mathcal{T}_R$ be the induced topology. Then:*

(a) *For all $w, v \in W$,*

$$(w, v) \in R \textit{ iff } (\forall U \in \mathcal{T}_R)[\, w \in U \Rightarrow v \in U \,]$$

(b) *For all $w \in W$,*

$$B_w = \bigcap \{U \in \mathcal{T}_R \mid w \in U\}$$

(c) *$\mathcal{T}_R$ is a D-topology.*

(d) *For $A \subseteq W$,*

$$int_{\mathcal{T}_R}(A) = \{w \in W \mid (\forall v \in W)[\textit{ if } (w, v) \in R \textit{ then } v \in A \,]\}$$

and

$$cl_{\mathcal{T}_R}(A) = \{w \in W \mid (\exists v \in W)[\, (w, v) \in R \textit{ and } v \in A \,]\}$$

Proof. Property (b) follows trivially from (a), and for (a), it suffices to observe that for all $w \in W$ and $U \in \mathcal{T}_R$, $w \in U$ iff $B_w \subseteq U$. Property (c) follows directly from (b). For (d), observe that for any $A \subseteq W$, the equivalence:

$$B_w \subseteq A \;\Leftrightarrow\; (\forall v \in W)[\text{ if } (w,v) \in R \text{ then } v \in A\,]$$

is trivial. So it suffices to show that $int_{\mathcal{T}_R}(A) = \{w \in W \mid B_w \subseteq A\}$. So suppose $w \in int_{\mathcal{T}_R}(A)$. Then for some $U \in \mathcal{T}_R$, $w \in U$ and $U \subseteq A$. Since the B_v's form a basis for $\mathcal{T}_R$, there is some $v \in W$ such that $w \in B_v$ and $B_v \subseteq U \subseteq A$. Then $B_w \subseteq B_v$ since $w \in B_v$, so we have $B_w \subseteq A$. Conversely, suppose $B_w \subseteq A$. Then taking $U = B_w$, we have a $U \in \mathcal{T}_R$ such that $w \in U$ and $U \subseteq A$. Hence $w \in int_{\mathcal{T}_R}(A)$. The closure equation comes by taking complements. ■

In [Grz67], Grzegorczyk uses an equivalent topology $\mathcal{T}'_R$ on W defined from R by:

$$cl_{\mathcal{T}'_R}(A) = \{w \in W \mid (\exists v \in W)[\,(w,v) \in R \text{ and } v \in A\,]\}$$

for all $A \subseteq W$, attributing it to [MT46]. This last equation can also be obtained from Jónsson and Tarski's work on Boolean algebra with operators [JT51].

Definition 2.4.9 *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a topological structure for $\mathcal{L}_{\Box a}$.*

- *$\mathfrak{T}$ is called an* D-topological structure *iff $\mathcal{T}$ is a D-topology on X.*
- *$\mathfrak{T}$ is called a* finite topological structure *iff the topology $\mathcal{T}$ is finite, i.e. $\mathcal{T}$ is a finite (complete) lattice of sets.*
- *$\mathfrak{T}$ is called a* finite-space topological structure *iff the space X is finite (and hence $\mathcal{T} \subseteq P(X)$ is finite).*

Trivially, finite-space topological structures are finite topological structures, and finite topological structures are D-topological structures. Finite-space topological structures correspond to finite Kripke frames.

We can now spell out the bijective transformation between Kripke frames and D-topological structures.

Proposition 2.4.10 [Grz67]. Duality of Kripke frames & D-topological structures

(i) *Given a Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$, let $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ be its induced D-topological structure. Then the Kripke frame $\mathcal{K}_{\mathfrak{T}_{\mathcal{K}}} = (W, R_{\mathcal{T}_R}, F)$ induced by $\mathfrak{T}_{\mathcal{K}}$ is such that:*

$$R = R_{\mathcal{T}_R}$$

(ii) *Given an D-topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for $\mathcal{L}_{\Box a}$, let $\mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$ be its induced Kripke frame. Then the D-topological structure $\mathfrak{T}_{\mathcal{K}_{\mathfrak{T}}} = (X, \mathcal{T}_{R_{\mathcal{T}}}, f)$ induced by $\mathcal{K}_{\mathfrak{T}}$ is such that:*

$$\mathcal{T} = \mathcal{T}_{R_{\mathcal{T}}}$$

Proof. Part (i) is an immediate consequence of Lemma 2.4.8, part (a). For (ii), let $\mathcal{B} = \{B_x \mid x \in X\}$ be the basis of $R_{\mathcal{T}}$-cones for the topology $\mathcal{T}_{R_{\mathcal{T}}}$ on X. Then for each $x \in X$,

$$B_x \doteq \{y \in X \mid (x, y) \in R_{\mathcal{T}}\} = \bigcap\{U \in \mathcal{T} \mid x \in U\}$$

by Proposition 2.4.6, part (c). Hence by part (a) of that same result, $\mathcal{B}$ is a basis for the D-topology $\mathcal{T}$ on X, so $\mathcal{T}_{R_{\mathcal{T}}} = \mathcal{T}$. ■

Thus the map

$$\mathcal{K} = (W, R, F) \mapsto \mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$$

from the class of Kripke frames for $\mathcal{L}_{\Box a}$ to the class of D-topological structures for $\mathcal{L}_{\Box a}$, is a bijection, with the map

$$\mathfrak{T} = (X, \mathcal{T}, f) \mapsto \mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$$

its inverse. The restrictions of the same maps establish a bijection between the class of *finite* Kripke frames for $\mathcal{L}_{\Box a}$ and the class of *finite-space* topological structures for $\mathcal{L}_{\Box a}$. The map also extends to a faithful bijection between valuations for D-topological structures and valuations for the corresponding Kripke frames.

Definition 2.4.11 Dual models *Given a Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$, let $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ be its induced D-topological structure. For each valuation $\eta : W \to \mathcal{P}(AP)$ for $\mathcal{K}$, define the* dual *of η to be the valuation $\xi_\eta : AP \to \mathcal{P}(W)$ for $\mathfrak{T}_{\mathcal{K}}$ given by*

$$w \in \xi_\eta(p) \quad \textit{iff} \quad p \in \eta(w)$$

for all $w \in W$ and $p \in AP$. The D-topological model $(\mathfrak{T}_{\mathcal{K}}, \xi_\eta)$ is called the dual *of the Kripke model $(\mathcal{K}, \eta)$.*

Similarly, given an D-topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for $\mathcal{L}_{\Box a}$, let $\mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$ be its induced Kripke frame. For each valuation $\xi : AP \to \mathcal{P}(X)$ for $\mathfrak{T}$, define the dual *of ξ to be the valuation $\eta_\xi : X \to \mathcal{P}(AP)$ for $\mathcal{K}_{\mathfrak{T}}$ given by*

$$p \in \eta_\xi(x) \quad \textit{iff} \quad x \in \xi(p)$$

for all $x \in X$ and $p \in AP$. The Kripke model $(\mathcal{K}_{\mathfrak{T}}, \eta_\xi)$ is called the dual *of the D-topological model $(\mathfrak{T}, \xi)$.*

Proposition 2.4.12 Duality of Kripke and D-topological models

(i) *Let* $(\mathcal{K}, \eta)$ *be a Kripke model for* $\mathcal{L}_{\Box a}$*, and* $(\mathfrak{T}_{\mathcal{K}}, \xi_\eta)$ *its dual D-topological model. Then for all worlds* w *of* $\mathcal{K}$ *and all formulas* φ *of* $\mathcal{L}_{\Box a}$*,*

$$w \in \|\varphi\|_{\xi_\eta} \quad \textit{iff} \quad w \Vdash_\eta \varphi$$

Hence

$$(\mathfrak{T}_{\mathcal{K}}, \xi_\eta) \models \varphi \quad \textit{iff} \quad (\mathcal{K}, \eta) \Vdash \varphi$$

(ii) *Let* $(\mathfrak{T}, \xi)$ *be an D-topological model for* $\mathcal{L}_{\Box a}$*, and* $(\mathcal{K}_{\mathfrak{T}}, \eta_\xi)$ *its dual Kripke model. Then for all states* x *of* $\mathfrak{T}$ *and all formulas* φ *of* $\mathcal{L}_{\Box a}$*,*

$$x \Vdash_{\eta_\xi} \varphi \quad \textit{iff} \quad x \in \|\varphi\|_\xi$$

Hence

$$(\mathcal{K}_{\mathfrak{T}}, \eta_\xi) \Vdash \varphi \quad \textit{iff} \quad (\mathfrak{T}, \xi) \models \varphi$$

Proof. For (i), the proof of

$$w \in \|\varphi\|_{\xi_\eta} \quad \Leftrightarrow \quad w \Vdash_\eta \varphi$$

is by induction on formulas. The base case for atomic formulas is immediate from the definition of the dual valuation ξ_η; the induction for Boolean connectives and $[a]$ is trivial; and for $\Box$, we use the interior operator equation from Lemma 2.4.8, part (d).

For (ii), the proof of

$$x \Vdash_{\eta_\xi} \varphi \quad \text{iff} \quad x \in \|\varphi\|_\xi$$

is essentially the same, using part (d) of Proposition 2.4.6 for the $\Box$ case of the induction. ■

Corollary 2.4.13 *For all formulas* φ *of* $\mathcal{L}_{\Box a}$*,*

$$\begin{array}{ll} & \mathfrak{T} \models \varphi \ \textit{ for all D-topological structures } \mathfrak{T} \textit{ for } \mathcal{L}_{\Box a} \\ \textit{iff} & \mathcal{K} \Vdash \varphi \ \textit{ for all Kripke frames } \mathcal{K} \textit{ for } \mathcal{L}_{\Box a} \end{array}$$

Corollary 2.4.14 *For all formulas* φ *of* $\mathcal{L}_{\Box a}$*,*

$$\begin{array}{ll} & \mathfrak{T} \models \varphi \ \textit{ for all } T_0 \textit{ D-topological structures } \mathfrak{T} \textit{ for } \mathcal{L}_{\Box a} \\ \textit{iff} & \mathcal{K} \Vdash \varphi \ \textit{ for all partially ordered Kripke frames } \mathcal{K} \textit{ for } \mathcal{L}_{\Box a} \end{array}$$

Chapter 3

The Logic S4C

3.1 Adding Continuity

In our definition of a topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for the language $\mathcal{L}_{\Box a}$, we place no restrictions on the function $f : X \to X$, other than totality. The language itself is rich enough to express various properties of f, notably the continuity of f with respect to the topology $\mathcal{T}$. The scheme

$$\textbf{Cont}: \ [a]\Box\varphi \to \Box[a]\varphi$$

is called the *continuity axiom*, in virtue of the following proposition.

Proposition 3.1.1 [Kur66] I,§13; [RS63] III,§3.
Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a topological structure for $\mathcal{L}_{\Box a}$. Then the following are equivalent:

(a) *for each $\varphi \in \mathcal{L}_{\Box a}$, $\mathfrak{T} \models [a]\Box\varphi \to \Box[a]\varphi$*

(b) *for each $\varphi \in \mathcal{L}_{\Box a}$, $\mathfrak{T} \models [a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi$*

(c) *the function $f : X \to X$ is continuous with respect to the topology $\mathcal{T}$.*

Proof. Let φ be any formula of $\mathcal{L}_{\Box a}$, let ξ be any valuation for $\mathfrak{T}$, and let $A = \|\varphi\|_\xi \subseteq X$. Then

$$\|[a]\Box\varphi \to \Box[a]\varphi\|_\xi = X \quad \text{iff} \quad f^{-1}(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

and

$$\|[a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi\|_\xi = X \quad \text{iff} \quad f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A)))$$

Now the following equivalence is immediate:

$$\begin{array}{lll} & (b): & f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \quad \text{for all } A \subseteq X \\ \text{iff} & (c): & f^{-1}(U) = int_{\mathcal{T}}(f^{-1}(U)) \quad \text{for all } U \in \mathcal{T} \\ & & \text{i.e. } f \text{ is continuous w.r.t. the topology } \mathcal{T} \end{array}$$

since $U \in \mathcal{T}$ iff $U = int_{\mathcal{T}}(U)$, and for any $A \subseteq X$, we have $int_{\mathcal{T}}(A) = U$ for some $U \in \mathcal{T}$. So rewriting

$$(a): \quad f^{-1}(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(f^{-1}(A)) \quad \text{for all } A \subseteq X$$

it suffices to show that (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

Assume (a) holds. Then for any $U \in \mathcal{T}$, we have $U = int_{\mathcal{T}}(U)$, hence

$$int_{\mathcal{T}}(f^{-1}(U)) \subseteq f^{-1}(U) = f^{-1}(int_{\mathcal{T}}(U)) \subseteq int_{\mathcal{T}}(f^{-1}(U))$$

and thus

$$f^{-1}(U) = int_{\mathcal{T}}(f^{-1}(U))$$

so (a) $\Rightarrow$ (c).

Now, for any $A \subseteq X$, we have $int_{\mathcal{T}}(A) \subseteq A$, hence applying $int_{\mathcal{T}} \circ f^{-1}$, we have

$$int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

Thus if (b) holds, we have

$$f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

hence (b) $\Rightarrow$ (a), as required. ■

The preceding proposition gives us an alternative, equivalent version of the continuity axiom, namely:

$$\mathbf{Cont}^*: \quad [a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi$$

It is also readily established that over the Hilbert system $\mathbf{S4F}_H$, the schemes **Cont** and **Cont*** are provably equivalent[1].

---

[1]The **Cont*** scheme is appealed to in devising a sequent calculus rule capturing continuity. The relevant rule is:

$$\frac{\Box[a]\Box\varphi, \Gamma \Rightarrow \Delta}{[a]\Box\varphi, \Gamma \Rightarrow \Delta}$$

which violates the sub-formula property, but in a managable way. Sequent calculi for **S4F** and **S4C** are investigated in [ADN97a].

From [RS63] and [Kur66], the converse of the **Cont** scheme,

$$\textbf{Open}: \quad \Box[a]\varphi \to [a]\Box\varphi$$

characterizes the *open mapping* property. All instances of the **Open** scheme are true in a topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$, exactly when the function $f : X \to X$ is such that for all $U \in \mathcal{T}$, the image $f(U) \in \mathcal{T}$, since the latter condition holds exactly when

$$int_{\mathcal{T}}(f^{-1}(A)) \subseteq f^{-1}(int_{\mathcal{T}}(A)) \quad \text{for all } A \subseteq X;$$

see [RS63], III,§3, p. 99, and [Kur66], I,§13,XIV. For total $f : X \to X$, the set map $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a *(topological) homomorphism* of the topological Boolean algebra $\mathfrak{B}_{\mathcal{T}}(X) = (\mathcal{P}(X), \cup, \cap, -, X, \emptyset, int_{\mathcal{T}})$ into itself, exactly when f is *both* continuous *and* open, since for such f we have:

$$\textbf{hom}: \quad f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(A)) \quad \text{for all } A \subseteq X$$

and f^{-1} commutes with all the Boolean operations. If in addition to **hom**, f^{-1} is both injective and surjective, then f^{-1} is an *automorphism* of the algebra $\mathfrak{B}_{\mathcal{T}}(X)$ onto itself; equivalently, f is a *homeomorphism* of X onto itself ([RS63], III,§3).

In this study, our chief interest is in continuity.

Definition 3.1.2 *Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box a}$. The map $F : W \to W$ is called R-*monotone *iff for all $w, v \in W$, $(w, v) \in R$ implies $(F(w), F(v)) \in R$.*

Proposition 3.1.3 Continuity in Kripke frames

(a) *Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box a}$, with $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ its induced D-topological structure.*

If F is R-monotone then F is continuous w.r.t. $\mathcal{T}_R$.

(b) *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a topological structure for $\mathcal{L}_{\Box a}$, with $\mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$ its induced Kripke frame.*

If f is continuous w.r.t. $\mathcal{T}$ then f is $R_{\mathcal{T}}$-monotone.

Proof. For (a), assume F is R-monotone. Then for arbitrary $A \subseteq W$ and $w \in W$,

$$\begin{array}{ll}
& w \in F^{-1}(int_{\mathcal{T}_R}(A)) \\
\Leftrightarrow & F(w) \in int_{\mathcal{T}_R}(A) \\
\Leftrightarrow & (\forall z \in W)[\,(F(w), z) \in R \Rightarrow z \in A\,] \\
\Rightarrow & (\forall v \in W)[\,(w, v) \in R \Rightarrow F(v) \in A\,] \qquad (*) \\
\Leftrightarrow & (\forall v \in W)[\,(w, v) \in R \Rightarrow v \in F^{-1}(A)\,] \\
\Leftrightarrow & w \in int_{\mathcal{T}_R}(F^{-1}(A))
\end{array}$$

with the implication $(*)$ a consequence of: $(w,v) \in R \Rightarrow (F(w), F(v)) \in R$. Hence $F^{-1}(int_{\mathcal{T}_R}(A)) \subseteq int_{\mathcal{T}_R}(F^{-1}(A))$, and so F is continuous with respect to $\mathcal{T}_R$.

For (b), assume f is continuous with respect to $\mathcal{T}$. Recall from Definition 2.4.1 that:

$$(x,y) \in R_{\mathcal{T}} \text{ iff } (\forall U \in \mathcal{T})[\, x \in U \Rightarrow y \in U \,]$$

Now fix $x, y \in X$, assume $(x,y) \in R_{\mathcal{T}}$, and let $U \in \mathcal{T}$ be any open set. Then

$$f(x) \in U \Leftrightarrow x \in f^{-1}(U) \Rightarrow y \in f^{-1}(U) \Leftrightarrow f(y) \in U$$

with the implication holding because $f^{-1}(U) \in \mathcal{T}$ (by continuity of f) and $(x,y) \in R_{\mathcal{T}}$. Hence $(f(x), f(y)) \in R_{\mathcal{T}}$, and so f is $R_{\mathcal{T}}$-monotone. ■

Part (b), the continuity of f w.r.t. $\mathcal{T}$ implying that f is *monotone* w.r.t. $R_{\mathcal{T}}$, is a variant of the theme of "continuity implies monotonicity" in cpo and domain theory. The result can be found in [Smy92], Proposition 4.2.4. It is part (a), while not deep, that is most pleasing. The two together give a particularly simple characterization of the meaning of continuity in a Kripke frame.

Definition 3.1.4 *A topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for $\mathcal{L}_{\Box_a}$ is called* continuous *iff f is continuous with respect to the topology $\mathcal{T}$. A Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box_a}$ is called* continuous *iff F is R-monotone.*

Proposition 3.1.5 Duality of continuous Kripke and D-topological models

(i) *Let $(\mathcal{K}, \eta)$ be a continuous Kripke model for $\mathcal{L}_{\Box_a}$, and $(\mathfrak{T}_{\mathcal{K}}, \xi_\eta)$ its dual continuous D-topological model. Then for all worlds w of $\mathcal{K}$ and all formulas φ of $\mathcal{L}_{\Box_a}$,*

$$w \in \|\varphi\|_{\xi_\eta} \quad \textit{iff} \quad w \Vdash_\eta \varphi$$

Hence

$$(\mathfrak{T}_{\mathcal{K}}, \xi_\eta) \models \varphi \quad \textit{iff} \quad (\mathcal{K}, \eta) \Vdash \varphi$$

(ii) *Let $(\mathfrak{T}, \xi)$ be a continuous D-topological model for $\mathcal{L}_{\Box_a}$, and $(\mathcal{K}_{\mathfrak{T}}, \eta_\xi)$ its dual continuous Kripke model. Then for all states x of $\mathfrak{T}$ and all formulas φ of $\mathcal{L}_{\Box_a}$,*

$$x \Vdash_{\eta_\xi} \varphi \quad \textit{iff} \quad x \in \|\varphi\|_\xi$$

Hence

$$(\mathcal{K}_{\mathfrak{T}}, \eta_\xi) \Vdash \varphi \quad \textit{iff} \quad (\mathfrak{T}, \xi) \models \varphi$$

Proof. Immediate from Proposition 3.1.3 together with Proposition 2.4.12. ■

Corollary 3.1.6 *For all formulas φ of $\mathcal{L}_{\Box a}$,*

$$\begin{array}{ll} & \mathfrak{T} \models \varphi \text{ for all continuous D-topological structures } \mathfrak{T} \text{ for } \mathcal{L}_{\Box a} \\ \textit{iff} & \mathcal{K} \Vdash \varphi \text{ for all continuous Kripke frames } \mathcal{K} \text{ for } \mathcal{L}_{\Box a} \end{array}$$

Corollary 3.1.7 *For all formulas φ of $\mathcal{L}_{\Box a}$,*

$$\begin{array}{ll} & \mathfrak{T} \models \varphi \text{ for all continuous } T_0 \text{ D-topological structures } \mathfrak{T} \text{ for } \mathcal{L}_{\Box a} \\ \textit{iff} & \mathcal{K} \Vdash \varphi \text{ for all continuous partially ordered Kripke frames } \mathcal{K} \text{ for } \mathcal{L}_{\Box a} \end{array}$$

It is readily verified that all instances of the scheme

$$\textbf{Open}: \quad \Box[a]\varphi \to [a]\Box\varphi$$

are forced in a Kripke frame $\mathcal{K} = (W, R, F)$ exactly when the condition

$$(F(w), u) \in R \;\Rightarrow\; (\exists v \in W)[\,F(v) = u \;\&\; (w, v) \in R\,] \qquad (F\text{-open})$$

holds for all $w, u \in W$. This condition is properly stronger than the converse of R-monotonicity:

$$(F(w), F(v)) \in R \;\Rightarrow\; (w, v) \in R$$

since the F-open condition can fail when F is not surjective; i.e. there is a $u \in W$ such that $u \neq F(v)$ for all $v \in W$. This is the case for the canonical term frame of a path through a tableaux in Section 4.2.

Observe that for a continuous Kripke frame $\mathcal{K} = (W, R, F)$, we always have

$$\begin{array}{rcll} F(B_w) & = & \{F(v) \mid (w, v) \in R\} & \\ & \subseteq & \{u \mid (F(w), u) \in R\} & \text{by the } R\text{-monotonicity of } F \\ & = & B_{F(w)} & \end{array}$$

and the inclusion will be strict whenever there is a $u \in W$ such that $(F(w), u) \in R$ but $u \neq F(v)$ for any v such that $(w, v) \in R$; i.e. when F fails to be an open map with respect to the cone topology $\mathcal{T}_R$. Note also that the only way $F(B_w)$ can be open

in $\mathcal{T}_R$ is if $F(B_w) = B_{F(w)}$, because $B_{F(w)}$ is the smallest open set in $\mathcal{T}_R$ containing $F(w)$, and $F(w) \in F(B_w)$. It then follows that:

$$
\begin{array}{ll}
& (\forall w, u \in W)\,[(w,v) \in R \Rightarrow (F(w), F(v)) \in R] \text{ and} \\
& (\forall w, u \in W)\,[(F(w), u) \in R \Rightarrow (\exists v \in W)[F(v) = u \;\&\; (w,v) \in R]] \\
\Leftrightarrow & \text{for all } w \in W,\ F(B_w) = B_{F(w)} \\
\Leftrightarrow & \text{for all } U \in \mathcal{T}_R,\ F^{-1}(U) \in \mathcal{T}_R \text{ and } F(U) \in \mathcal{T}_R \\
\Leftrightarrow & F \text{ is a } \textit{continuous} \text{ and } \textit{open} \text{ map w.r.t. } \mathcal{T}_R \\
\Leftrightarrow & \mathfrak{T}_{\mathcal{K}} \models [a]\Box\varphi \leftrightarrow \Box[a]\varphi \quad \text{for all formulas } \varphi \text{ of } \mathcal{L}_{\Box a} \\
\Leftrightarrow & \mathcal{K} \Vdash [a]\Box\varphi \leftrightarrow \Box[a]\varphi \quad \text{for all formulas } \varphi \text{ of } \mathcal{L}_{\Box a}
\end{array}
$$

3.2 Hilbert-style Proof System

Definition 3.2.1 *The Hilbert-style proof system for the logic* **S4C** *has as its axiom schemes those of* **S4F** *(Definition 2.2.1) together with all instances of the scheme*

$$\textbf{Cont}: \quad [a]\Box\varphi \to \Box[a]\varphi$$

in the language $\mathcal{L}_{\Box a}$; the inference rules are the same as those of **S4F**.

We write

$$\textbf{S4C} \vdash_H \varphi$$

or say φ is **S4C**$_H$ *provable, if the formula $\varphi \in \mathcal{L}_{\Box a}$ has an* **S4C** *Hilbert-style derivation.*

The following are derivable in **S4C**$_H$, for any formula $\varphi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$.

$$
\begin{array}{ll}
[a]^k\textbf{Cont}: & [a]^k\Box\varphi \to \Box[a]^k\varphi \\
[a]^k\Diamond\textbf{Cont}: & \Diamond[a]^k\varphi \to [a]^k\Diamond\varphi
\end{array}
$$

The following is an admissible inference rule in **S4C**$_H$, for any formulas $\varphi, \psi, \chi \in \mathcal{L}_{\Box a}$ and $k, l \in \mathbb{N}$:

$$
\begin{array}{ll}
\textit{Continuous} & \\
\textit{Hoare composition:} & \dfrac{\varphi \to [a]^k\Box\chi,\ \chi \to [a]^l\Box\psi}{\varphi \to [a]^{k+l}\Box\psi}
\end{array}
$$

Proposition 3.2.2 Soundness of **S4C** Hilbert-style proof system
For all formulas φ of $\mathcal{L}_{\Box a}$, if $\mathbf{S4C} \vdash_H \varphi$,
then $\mathfrak{T} \models \varphi$ *for all continuous topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$
and $\mathcal{K} \Vdash \varphi$ *for all continuous Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$.

Proof. Immediate from Propositions 2.2.2, 3.1.1 and 3.1.6. ■

As for **S4F**, a "cheap" proof of Kripke completeness for the **S4C** Hilbert-style proof system is available.

Proposition 3.2.3 Kripke Completeness of **S4C** Hilbert-style proof system
There exists a continuous Kripke model $(\mathcal{K}_0, \eta_0)$ *such that for all formulas* φ *of* $\mathcal{L}_{\Box a}$,

$$(\mathcal{K}_0, \eta_0) \Vdash \varphi \quad \textit{iff} \quad \mathbf{S4C} \vdash_H \varphi$$

Proof. From the proof of Proposition 2.3.7, the canonical Kripke frame $\mathcal{K}_0 = (W_0, R_0, F_0)$ satisfies:

$$\begin{aligned} W_0 &= \{U \subseteq \mathcal{L}_{\Box a} \mid U \text{ is maximal } \mathbf{S4C}\text{-consistent}\} \\ (U,V) \in R_0 &\text{ iff } (\forall \varphi \in \mathcal{L}_{\Box a})[\,\Box\varphi \in U \Rightarrow \varphi \in V\,] \\ F_0(U) &= \{\varphi \in \mathcal{L}_{\Box a} \mid [a]\varphi \in U\} \end{aligned}$$

with the canonical valuation η_0 for $\mathcal{K}_0$ given by: $p \in \eta_0(U)$ iff $p \in U$. It suffices to show that the function F_0, which "peels-off" one $[a]$, is monotone with respect to the relation R_0. Fix maximal **S4C**-consistent sets U, V and a formula φ, suppose $(U,V) \in R_0$. Then

$$\begin{array}{lll} & \Box\varphi \in F_0(U) & \\ \Leftrightarrow & [a]\Box\varphi \in U & \text{property of } F_0 \\ \Rightarrow & \Box[a]\varphi \in U & \textbf{Cont} \text{ axiom} \\ \Rightarrow & [a]\varphi \in V & \text{definition of } R_0 \\ \Leftrightarrow & \varphi \in F_0(V) & \text{property of } F_0 \end{array}$$

Hence $(F_0(U), F_0(V)) \in R_0$, as required. ■

3.3 Quotient Kripke Frames and T_0 Topologies

This section examines the T_0 quotient construction, and the dual construction of a partially-ordered Kripke frame. We identify a class of valuations which can be faithfully passed through to T_0 quotients. The first task is to formalize the notion of a quotient of a Kripke frame in our setting.

Definition 3.3.1 *Let $\mathcal{K} = (W, R, F)$ and $\mathcal{K}' = (W', R', F')$ be Kripke frames for $\mathcal{L}_{\Box a}$, and let $h : W \to W'$ be a surjective map.*

We say the frame $\mathcal{K}'$ is a quotient under h of the frame $\mathcal{K}$ *iff*

(a) *h preserves the accessibility relations R and R': for all $w, v \in W$,*

$$(w, v) \in R \quad \Rightarrow \quad (h(w), h(v)) \in R'$$

(b) *h preserves the functions F and F':*

$$F' \circ h = h \circ F$$

$\mathcal{K}'$ is called the minimal quotient under h of $\mathcal{K}$ *iff for all $w, v \in W$,*

$$(h(w), h(v)) \in R' \quad \Leftrightarrow \quad (w, v) \in R$$

Lemma 3.3.2 *Given a Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$ and a surjective map $h : W \to W'$ onto a non-empty set W', any structure $\mathcal{K}' = (W', R', F')$ satisfying:*

(1) *$R' \subseteq W' \times W'$ is such that for all $w, v \in W$,*

$$(h(w), h(v)) \in R' \quad \Leftrightarrow \quad (w, v) \in R$$

(2) *$F' : W' \to W'$ is such that for all $w \in W$,*

$$F'(h(w)) = h(F(w))$$

is a Kripke frame for $\mathcal{L}_{\Box a}$, and the minimal quotient under h of $\mathcal{K}$.
Moreover, if $\mathcal{K}$ is a continuous Kripke frame, then $\mathcal{K}'$ is also continuous.

Proof. Clause (1) guarantees that R' is a reflexive and transitive binary relation on W', and by the surjectivity of h, the equation in clause (2) defines a total function F' on W'. If F is R-monotone, then for all $w, v \in W$,

$$\begin{array}{lll} & (h(w), h(v)) \in R' & \\ \Leftrightarrow & (w, v) \in R & (1) \\ \Rightarrow & (F(w), F(v)) \in R & \text{monotonicity} \\ \Leftrightarrow & (h(F(w)), h(F(v))) \in R' & (1) \\ \Leftrightarrow & (F'(h(w)), F'(h(v))) \in R' & (2) \end{array}$$

Hence F' is R'-monotone. ■

Recall from Lemma 2.4.8, part (a), that in a Kripke frame $\mathcal{K} = (W, R, F)$, with its induced topology $\mathcal{T}_R$ on W,

$$(w, v) \in R \text{ and } (v, w) \in R \quad \text{iff} \quad (\forall U \in \mathcal{T}_R)[\, w \in U \;\Leftrightarrow\; v \in U \,].$$

i.e. w and v belong to all the same open sets in $\mathcal{T}_R$. So we call the quotient which identifies all such w and v the T_0 quotient of $\mathcal{K}$.

Proposition 3.3.3 *Let $\mathcal{K} = (W, R, F)$ be a continuous Kripke frame for $\mathcal{L}_{\Box a}$. For each $w \in W$, let:*

$$\bar{w} = \{v \in W \mid (w, v) \in R \textit{ and } (v, w) \in R\}$$

Let $W^0 = \{\bar{w} \mid w \in W\}$, and let $h : W \to W^0$ be the surjective map given by $h(w) = \bar{w}$.

Then the structure $\mathcal{K}^0 = (W^0, R^0, F^0)$ defined, for all $w, v \in W$, by:

(i) $(h(w), h(v)) \in R^0$ *iff* $(w, v) \in R$*; and*

(ii) $F^0(h(w)) = h(F(w))$

is a Kripke frame, and $\mathcal{K}^0$ is the minimal quotient under h_R of $\mathcal{K}$.

Moreover, $\mathcal{K}^0$ is a continuous Kripke frame, and the relation R^0 is a partial order, so its induced topology $\mathcal{T}_{R^0}$ on W^0 is T_0. The frame $\mathcal{K}^0$ will be called the T_0 quotient *of $\mathcal{K}$.*

Proof. First observe that the R-monotonicity of F is what is needed to ensure that the function F^0 is well-defined by (ii) (i.e. this quotient construction is not available for arbitrary Kripke frames), and F^0 is R^0-monotone by Lemma 3.3.2. The anti-symmetry of the relation R^0 is immediate from the definition of the quotient map h. ■

We now turn to the topological quotient.

Definition 3.3.4 *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a continuous topological structure for $\mathcal{L}_{\Box a}$. Define an equivalence relation $\equiv_0$ on X by:*

$$x \equiv_0 y \quad \textit{iff} \quad (\forall U \in \mathcal{T})[\, x \in U \Leftrightarrow y \in U \,];$$

equivalently,

$$x \equiv_0 y \quad \textit{iff} \quad (x, y) \in R_{\mathcal{T}} \;\; \textit{and} \;\; (y, x) \in R_{\mathcal{T}}$$

where $R_{\mathcal{T}}$ is the relation induced by $\mathcal{T}$. Let $\bar{x} = \{y \in X \mid x \equiv_0 y\}$ denote the equivalence class of x under $\equiv_0$, and let $X^0 = \{\bar{x} \mid x \in X\}$ be the set of equivalence

classes. Let $q : X \to X^0$ be the surjective map given by $q(x) = \bar{x}$. The map q is called the Stone map *for $\mathfrak{T}$. Then q induces a unique quotient topology $\mathcal{T}^0$ on X^0: for subsets $V \subseteq X^0$,*

$$V \in \mathcal{T}^0 \quad \textit{iff} \quad q^{-1}(V) \in \mathcal{T}$$

By standard arguments (see, for example [Th66], *Theorem 14.2), $\mathcal{T}^0$ is a T_0 topology on X^0 and the Stone map $q : X \to X^0$ is both open and closed, as well as (trivially) continuous. Define $f^0 : X^0 \to X^0$ to be the unique map satisfying*

$$f^0 \circ q = q \circ f$$

Then (by standard arguments) f^0 is well-defined on the quotient and continuous with respect to the quotient topology $\mathcal{T}^0$. Hence $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ is a T_0 continuous topological structure, called the T_0 quotient *of $\mathfrak{T}$.*

The Kripke frame and topological T_0 quotient constructions clearly commute with the duality transformations $\mathcal{K} \mapsto \mathfrak{T}_{\mathcal{K}}$ and $\mathfrak{T} \mapsto \mathcal{K}_{\mathfrak{T}}$ between continuous Kripke frames and continuous D-topological structures.

Proposition 3.3.5 *Let $\mathcal{K} = (W, R, F)$ be a continuous Kripke frame for $\mathcal{L}_{\Box a}$, with $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ its induced D-topological structure, and $\mathcal{K}^0 = (W^0, R^0, F^0)$ its T_0 quotient.*

Then $\mathfrak{T}^0_{\mathcal{K}}$, the T_0 quotient of $\mathfrak{T}_{\mathcal{K}}$, and $\mathfrak{T}_{\mathcal{K}^0}$, the induced D-topological structure of $\mathcal{K}^0$, are identical *topological structures.*

Dually, let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a continuous D-topological structure, with $\mathcal{K}_{\mathfrak{T}} = (X, R_{\mathcal{T}}, f)$ its induced Kripke frame, and $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ its T_0 quotient.

Then $\mathcal{K}^0_{\mathfrak{T}}$, the T_0 quotient of $\mathcal{K}_{\mathfrak{T}}$, and $\mathcal{K}_{\mathfrak{T}^0}$, the induced Kripke frame of $\mathfrak{T}^0$, are identical *Kripke frames.*

Proof. Immediate. ■

Corollary 3.3.6 *If $\mathfrak{T} = (X, \mathcal{T}, f)$ is a continuous D-topological structure for $\mathcal{L}_{\Box a}$ and $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ is its T_0 quotient, then $\mathcal{T}^0$ is a D-topology.*

Having established the structure of topological and Kripke T_0 quotients, we turn to the question of how and when a *valuation* of atomic propositions in a topological or Kripke model can be faithfully passed through to the corresponding T_0 quotient. We start with a lemma on how the quotient map behaves with the interior and inverse-image operators.

Lemma 3.3.7 *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a continuous topological structure for $\mathcal{L}_{\Box a}$ and let $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ be its T_0 quotient, with $q : X \to X^0$ the Stone map. Then for all $A \subseteq X^0$,*

$$\begin{aligned} q^{-1}\left((f^0)^{-1}(A)\right) &= f^{-1}\left(q^{-1}(A)\right) \\ q^{-1}\left(int_{\mathcal{T}^0}(A)\right) &= int_{\mathcal{T}}\left(q^{-1}(A)\right) \end{aligned}$$

Proof. The first formula is immediate from the defining equation for f^0, namely $f^0 \circ q = q \circ f$. For the second formula, the inclusion

$$q^{-1}\left(int_{\mathcal{T}^0}(A)\right) \subseteq int_{\mathcal{T}}\left(q^{-1}(A)\right)$$

for $A \subseteq X^0$, is derivable using only the fact that q^{-1} commutes with arbitrary unions and is inclusion monotone, together with the defining property of the quotient topology: $V \in \mathcal{T}^0$ iff $q^{-1}(V) \in \mathcal{T}$, for $V \subseteq X^0$. For the converse inclusion of the second formula, observe that the subset $q\left(int_{\mathcal{T}}\left(q^{-1}(A)\right)\right)$ of X^0 is open w.r.t. $\mathcal{T}^0$, since $q : X \to X^0$ is an open map and $int_{\mathcal{T}}\left(q^{-1}(A)\right)$ is trivially open w.r.t. $\mathcal{T}$. So to prove:

$$q\left(int_{\mathcal{T}}\left(q^{-1}(A)\right)\right) \subseteq int_{\mathcal{T}^0}(A) \quad (\#)$$

it suffices to show:

$$q\left(int_{\mathcal{T}}\left(q^{-1}(A)\right)\right) \subseteq A \quad (*)$$

and (*) is readily verified using basic properties of $int_{\mathcal{T}}$ and q^{-1}. The desired inclusion then follows from (#) by applying q^{-1} to both sides. ■

Given $\mathfrak{T} = (X, \mathcal{T}, f)$ continuous, with $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ its T_0 quotient and $q : X \to X^0$ the Stone map, the preceding lemma suggests that we try to pass a valuation $\xi : AP \to \mathcal{P}(X)$ through to the quotient by defining $\xi^0 : AP \to \mathcal{P}(X^0)$ by:

$$q(x) \in \xi^0(p) \quad \text{iff} \quad x \in \xi(p)$$

for all $p \in AP$ and $x \in X$. *Provided* ξ^0 is well-defined, we then get the base case of an induction proving:

$$q^{-1}\left(\|\varphi\|_{\xi^0}\right) = \|\varphi\|_{\xi}$$

The well-definedness of ξ^0 requires that if $q(x) = q(y)$ then for all $p \in AP$, $x \in \xi(p)$ iff $y \in \xi(p)$. Since $q(x) = q(y)$ iff $(x, y) \in R_{\mathcal{T}}$ and $(y, x) \in R_{\mathcal{T}}$, well-definedness can be characterized in terms of the induced relation $R_{\mathcal{T}}$.

Definition 3.3.8 T_0-consistent valuations

Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a continuous topological structure for $\mathcal{L}_{\Box a}$, with $R_{\mathcal{T}}$ the relation induced by $\mathcal{T}$. A valuation $\xi : AP \to \mathcal{P}(X)$ for $\mathfrak{T}$ is called T_0-consistent *iff for all $x, y \in X$,*

$$(x, y) \in R_{\mathcal{T}} \quad \textit{implies} \quad (\forall p \in AP)[\, x \in \xi(p) \Rightarrow y \in \xi(p)\,]$$

Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box a}$. A valuation $\eta : W \to \mathcal{P}(AP)$ for $\mathcal{K}$ is called T_0-consistent *iff for all $w, v \in W$,*

$$(w, v) \in R \quad \textit{implies} \quad \eta(w) \subseteq \eta(v)$$

i.e. η is inclusion-monotone with respect to R.

So the T_0-consistency condition for valuations is essentially *monotonicity*, reminiscent of the monotonicity requirement for valuations in the (partial-order) Kripke semantics for Intuitionistic logic. Observe that if ξ is an *open* valuation for $\mathfrak{T}$, i.e. $\xi(p) \in \mathcal{T}$ for all $p \in AP$, then ξ is T_0-consistent.

Theorem 3.3.9 *Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a continuous topological structure for $\mathcal{L}_{\Box a}$, let $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ be its T_0 quotient, with $q : X \to X^0$ the Stone map, and let $\xi : AP \to \mathcal{P}(X)$ be a T_0-consistent valuation for $\mathfrak{T}$.*

Define a valuation $\xi^0 : AP \to \mathcal{P}(X^0)$ for $\mathfrak{T}^0$ by:

$$q(x) \in \xi^0(p) \quad \textit{iff} \quad x \in \xi(p)$$

for all $p \in AP$ and $x \in X$,

Then for all $\varphi \in \mathcal{L}_{\Box a}$,

$$q^{-1}\left(\|\varphi\|_{\xi^0}\right) = \|\varphi\|_{\xi}$$

Hence

$$(\mathfrak{T}^0, \xi^0) \models \varphi \quad \textit{iff} \quad (\mathfrak{T}, \xi) \models \varphi$$

Proof. Proceed by induction on formulas φ of $\mathcal{L}_{\Box a}$. The T_0-consistency condition guarantees that the valuation ξ^0 is well-defined, in which case

$$q^{-1}\left(\xi^0(p)\right) = \xi(p)$$

for atomic propositions $p \in AP$. For the induction cases of $\neg$, $\to$, $\Box$ and $[a]$, respectively, we use the formulas:

$$\begin{aligned}
q^{-1}(-A\,) &= -q^{-1}(A) \\
q^{-1}(-A \cup B) &= -q^{-1}(A) \cup q^{-1}(B) \\
q^{-1}(int_{\mathcal{T}^0}(A)) &= int_{\mathcal{T}}(q^{-1}(A)) \\
q^{-1}((f^0)^{-1}(A)) &= f^{-1}(q^{-1}(A))
\end{aligned}$$

with the latter two coming from Lemma 3.3.7.

Then since

$$\|\varphi\|_{\xi^0} = X^0 \quad \Leftrightarrow \quad q^{-1}\left(\|\varphi\|_{\xi^0}\right) = X$$

with the ($\Rightarrow$) direction from the totality of q, and the ($\Leftarrow$) direction by the surjectivity of q, we have

$$(\mathfrak{T}^0, \xi^0) \models \varphi \quad \text{iff} \quad (\mathfrak{T}, \xi) \models \varphi$$

■

Corollary 3.3.10 *Let $\mathcal{K} = (W, R, F)$ be a continuous Kripke frame for $\mathcal{L}_{\Box a}$, let $\mathcal{K}^0 = (W^0, R^0, F^0)$ be its T_0 quotient, with $h : W \to W^0$ the quotient map, and let $\eta : W \to \mathcal{P}(AP)$ be a T_0-consistent valuation for $\mathcal{K}$.*

Define a valuation $\eta^0 : W^0 \to \mathcal{P}(AP)$ for $\mathcal{K}^0$ by:

$$\eta^0(h(w)) = \eta(w)$$

for all $w \in W$.

Then for all $\varphi \in \mathcal{L}_{\Box a}$ and all $w \in W$,

$$h(w) \Vdash_{\eta^0} \varphi \quad \textit{iff} \quad w \Vdash_{\eta} \varphi$$

Hence

$$(\mathcal{K}^0, \eta^0) \Vdash \varphi \quad \textit{iff} \quad (\mathcal{K}, \eta) \Vdash \varphi$$

Proof. The T_0-consistency of $\eta : W \to \mathcal{P}(AP)$ implies that its dual valuation $\xi_\eta : AP \to \mathcal{P}(W)$ for the induced topological structure $\mathfrak{T}_{\mathcal{K}}$ is T_0-consistent, using the fact that $R_{\mathcal{T}_R} = R$ from Proposition 2.4.10. Then apply Theorem 3.3.9, together with Propositions 3.3.5 and 3.1.5. ■

It is clear that an *arbitrary* valuation for a continuous topological structure or Kripke frame cannot be passed faithfully through to the T_0 quotient. From Kripke's [Kri63], the formula:

$$\Diamond(\Box p \vee \Box\neg p)$$

is *not* **S4** provable, but it is true in every **S4** Kripke frame $\mathcal{K} = (W, R)$ with the property that for each $w \in W$, there is a *terminal* world v such that $(w, v) \in R$. A world v is terminal if $(v, u) \in R$ iff $u = v$. Every $\mathcal{K} = (W, R)$ with W finite and R a partial order has the "access to a terminal world" property. In Section 4.5, we will establish that **S4C** is complete for the class of finite continuous Kripke frames for $\mathcal{L}_{\Box a}$, but applying Kripke's observation, **S4C** cannot be complete for the class of finite continuous partially ordered Kripke frames for $\mathcal{L}_{\Box a}$.

Recall from Section 1.3 that the natural analog-to-digital conversion map for a D-space is the Stone T_0 quotient. Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a structure with f continuous and $\mathcal{T}$ a *finite* D-topology. Let $\{B_i\}_{i \in I}$, where $I = \{1, ..., n\}$, be the basis for $\mathcal{T}$ obtained by taking all the non-empty join-irreducibles in the (finite) lattice $\mathcal{T}$, so that for each $i \in I$, there is an $x \in X$ such that:

$$B_i = B_x \doteq \bigcap\{U \in \mathcal{T} \mid x \in U\}$$

and for each $x \in X$, $B_x = B_i$ for some $i \in I$. Then the analog-to-digital conversion map $AD : X \to I$ is given by:

$$AD(x) = i \quad \text{iff} \quad B_i = B_x$$

Let $\mathfrak{T}^0 = (X^0, \mathcal{T}^0, f^0)$ be the T_0 quotient of $\mathfrak{T}$, with $q : X \to X^0$ the Stone map. The mapping $d : X^0 \to I$ given by $d(q(x)) = AD(x)$ is well-defined, since:

$$q(x) = q(y) \quad \text{iff} \quad (\forall U \in \mathcal{T})[\, x \in U \Leftrightarrow y \in U\,];$$

moreover, it establishes a bijection between X^0 and I. If I is partially ordered by: $i \leq j$ iff $B_j \subseteq B_i$, and then given the Alexandroff topology $\mathcal{T}_{\leq}$ generated by the upper cones under $\leq$, it is readily verified that d is a homeomorphism.

Theorem 3.3.9 says that if we want satisfiability and truth in models $(X, \mathcal{T}, f; \xi)$ to pass faithfully to the finite T_0 quotient $(X^0, \mathcal{T}^0, f^0; \xi^0)$ under the AD map, we must be careful with the choice of valuations ξ. In practice, this is not a problem, since atomic propositions will typically be evaluated by the basic open sets B_i, and as noted above, any open valuation is T_0 consistent.

Chapter 4

Tableaux Proof Systems

4.1 S4F and S4C Tableaux

In this chapter, we give a detailed presentation of a tableaux proof system for the logics **S4F** and **S4C**. The system is an extension of the treatment of modal tableaux in [NS93] and [Ne90], which is in turn a descendant of the modal *prefixed tableaux* systems of Fitting [Fi72] and [Fi83] Ch. 8. The essential idea, which traces back to Fitch, is to add to the *formal language of proofs* symbols intended to name possible *worlds* in Kripke models, taking to heart the central idea of Beth [Be59] that the construction of a tableaux proof is an attempt to build a countermodel. So to give symbolic representation to such models, I include in the formal language of proofs not only symbols for possible worlds, but also symbols for both the accessibility *relation* and the *function*.

A tableaux is a labeled binary tree, where the labels, called *entries*, are of two sorts:

- *signed forcing assertions* $T[\, t \Vdash \varphi \,]$ or $F[\, t \Vdash \varphi \,]$, and
- *modal accessibility assertions* $t\mathbf{R}s$,

where the terms t, s are functional terms generated from a set of primitive world symbols by applying the unary function symbol $\mathbf{F}$.
The root entry of a tableaux will always be a signed forcing assertion in which the term t is required to be a primitive world symbol $\mathbf{w}$. The tableaux construction rules, for extending a path in a tree, correspondingly represent two sorts of inference:

- *rules for the logical analysis of signed forcing assertions* $T[\, t \Vdash \varphi \,]$ or $F[\, t \Vdash \varphi \,]$, in terms of the principle connective or modal operator of the formula φ, designed to capture the various clauses of the inductive definition of forcing, and

- *rules for the accessibility relation*, capturing the reflexivity and transitivity of $\mathbf{R}$, and for the class of **S4C** tableaux, a rule capturing the monotonicity of $\mathbf{F}$ with respect to $\mathbf{R}$.

In the course of constructing a tableaux, complex formulas φ are broken down while complex terms t are built up.

A formula φ has a tableaux proof exactly when all paths through a tableaux with root $F[\ \mathrm{w}_0 \Vdash \varphi\ ]$ are contradictory, while a non-contradictory path P through a "finished" tableaux with root $F[\ \mathrm{w}_0 \Vdash \varphi\ ]$ gives us the symbolic material to construct explicitly a Kripke frame $\mathcal{K}_{\mathsf{P}}$ and a valuation η_{P} such that $\mathrm{w}_0 \nVdash_{\eta_{\mathsf{P}}} \varphi$. Symmetrically, from a non-contradictory path P through a "finished" tableaux with root $T[\ \mathrm{w}_0 \Vdash \varphi\ ]$, we can show that in the Kripke model $(\mathcal{K}_{\mathsf{P}}, \eta_{\mathsf{P}})$, we have $\mathrm{w}_0 \Vdash_{\eta_{\mathsf{P}}} \varphi$.

Definition 4.1.1 *Let* $\mathbf{W}$ *be a countable set of* world symbols *and let* $\mathbf{F}$ *be a unary function symbol. Let* $\mathbf{W}(\mathbf{F})$ *be the set of terms generated from* $\mathbf{W}$ *under* $\mathbf{F}$*; i.e. the free term algebra. A* $t \in \mathbf{W}(\mathbf{F})$ *is called a* world term. *So every world term* $t \in \mathbf{W}(\mathbf{F})$ *is of the form* $\mathbf{F}^k(\mathrm{w})$ *for some world symbol* $\mathrm{w} \in \mathbf{W}$ *and integer* $k \geq 0$*, where* $\mathbf{F}^0(\mathrm{w})$ *is* w. *Terms* $\mathbf{F}^k(\mathrm{w})$ *for* $k > 0$ *are referred to as* complex world terms, *while world symbols* $\mathrm{w} \in \mathbf{W}$ *are called* simple world terms.

Definition 4.1.2 *For a signed forcing assertion of the form:*

$$T[\ t \Vdash \varphi\ ] \quad or \quad F[\ t \Vdash \varphi\ ]$$

the world term t is said to be the subject *of the assertion, and the formula* φ *is said to be the* object *of the assertion.*

If $t = \mathbf{F}^k(\mathrm{w}_i)$ *then* t *is said to be* relevant *to any signed forcing assertion which has* w_i *as its subject.*

We assume we have a fixed enumeration $\{\mathrm{w}_i \mid i \in \mathbb{N}\}$ of $\mathbf{W}$. In the section on completeness, we will also have need of a fixed enumeration of all the world terms in $\mathbf{W}(\mathbf{F})$, where set-wise $\mathbf{W}(\mathbf{F}) \cong \mathbb{N} \times \mathbb{N}$. Let $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the standard bijective pairing function

$$p(i,k) = i + \left(\sum_{n<i+k} (n+1) \right) = \frac{1}{2}\left((i+k)^2 + 3i + k\right)$$

We then take

$$\{s_j \mid i,k,j \in \mathbb{N} \text{ and } j = p(i,k) \text{ and } s_j = \mathbf{F}^k(\mathrm{w}_i)\}$$

as our enumeration of $\mathbf{W}(\mathbf{F})$. We also have need for well-ordered index sets. We write $I \preceq \mathbb{N}$ to mean I is either a non-empty finite initial segment of $\mathbb{N}$ or else all of $\mathbb{N}$, and we write $\emptyset \preceq I \preceq \mathbb{N}$ to allow the possibility that I is empty.

Definition 4.1.3 *The following labeled binary trees are* atomic tableaux, *where* $p \in AP$, $\varphi, \psi \in \mathcal{L}_{\Box a}$, *and* $t, s \in \mathbf{W}(\mathbf{F})$.

$$(T\text{-}AP) \quad T[\, t \Vdash p \,] \qquad\qquad (F\text{-}AP) \quad F[\, t \Vdash p \,]$$

$$(T\neg) \quad \begin{array}{c} T[\, t \Vdash \neg\varphi \,] \\ | \\ F[\, t \Vdash \varphi \,] \end{array} \qquad\qquad (F\neg) \quad \begin{array}{c} F[\, t \Vdash \neg\varphi \,] \\ | \\ T[\, t \Vdash \varphi \,] \end{array}$$

$$(T \rightarrow) \quad \begin{array}{ccc} & T[\, t \Vdash \varphi \rightarrow \psi \,] & \\ / & & \backslash \\ F[\, t \Vdash \varphi \,] & & T[\, t \Vdash \psi \,] \end{array} \qquad\qquad (F \rightarrow) \quad \begin{array}{c} F[\, t \Vdash \varphi \rightarrow \psi \,] \\ | \\ T[\, t \Vdash \varphi \,] \\ | \\ F[\, t \Vdash \psi \,] \end{array}$$

$$(T\Box) \quad \begin{array}{c} T[\, t \Vdash \Box\varphi \,] \\ | \\ T[\, s \Vdash \varphi \,] \end{array} \qquad\qquad (F\Box) \quad \begin{array}{c} F[\, t \Vdash \Box\varphi \,] \\ | \\ t\mathbf{R}\mathbf{w}_j \\ | \\ F[\, \mathbf{w}_j \Vdash \varphi \,] \end{array} \quad \mathbf{w}_j \in \mathbf{W} \textit{ new}$$

if $t\mathbf{R}s$ *occurs previously on this path*

$$(T[a]) \quad \begin{array}{c} T[\, t \Vdash [a]\varphi \,] \\ | \\ T[\, \mathbf{F}(t) \Vdash \varphi \,] \end{array} \qquad\qquad (F[a]) \quad \begin{array}{c} F[\, t \Vdash [a]\varphi \,] \\ | \\ F[\, \mathbf{F}(t) \Vdash \varphi \,] \end{array}$$

Definition 4.1.4 *The class of* **S4F** tableaux *is defined inductively as follows:*

(i) *If τ is an atomic tableaux in which the world term t in the root entry is a world symbol $\mathbf{w}_i \in \mathbf{W}$, then τ is an* **S4F** *tableaux.*

For the case $(F\Box)$, the condition that the $\mathbf{w}_j$ in $\mathbf{w}_i\mathbf{R}\mathbf{w}_j$ be "new" merely means that $j \neq i$; for definiteness, we may take $j = i + 1$.

For the case $(T\Box)$, the condition that $t\mathbf{R}s$ "occurs previously on this path" cannot be satisfied in this case, so an atomic tableaux τ with root entry $T[\ \mathbf{w}_i \Vdash \Box\varphi\ ]$ consists of the root node only.

(ii) *If τ is a finite* **S4F** *tableaux, P is a path in τ which does* not *contain contradictory entries:*

$$T[\ t \Vdash \varphi\ ] \quad and \quad F[\ t \Vdash \varphi\ ]$$

for any $\varphi \in \mathcal{L}_{\Box a}$ and $t \in \mathbf{W}(\mathbf{F})$, and τ' is constructed from τ by extending P using one of the following construction rules, then τ' is an **S4F** *tableaux.*

(Develop) *A signed forcing assertion E occurs on P and τ' is constructed from τ by appending an atomic tableaux with root entry E to the end of the path P.*

For the case $(F\Box)$, where E is of the form $F[\ t \Vdash \Box\varphi\ ]$, the condition that the $\mathbf{w}_j$ in $t\mathbf{R}\mathbf{w}_j$ be "new" means that $j \in \mathbb{N}$ is the least integer such that $\mathbf{w}_j$ is yet to occur in any entry on τ.

For the case $(T\Box)$, where E is of the form $T[\ t \Vdash \Box\varphi\ ]$, the condition that $t\mathbf{R}s$ "occurs previously on this path" means that $t\mathbf{R}s$ is an entry on P. If there are no entries $t\mathbf{R}s$ on P, for any $s \in \mathbf{W}(\mathbf{F})$, then as in (i)*, the atomic tableaux in this case consists only of the root node labelled E.*

(R-Reflex) *A world term $t \in \mathbf{W}(\mathbf{F})$ is relevant to some signed forcing assertion on P, and τ' is constructed from τ by adjoining to the end of P an entry $t\mathbf{R}t$.*

(R-Trans) *For some $t, s, r \in \mathbf{W}(\mathbf{F})$, accessibility assertions $t\mathbf{R}r$ and $r\mathbf{R}s$ both occur as entries on P, and τ' is constructed from τ by adjoining to the end of P the entry $t\mathbf{R}s$.*

(iii) *If $I \preceq \mathbb{N}$ and $\{\tau_n\}_{n\in I}$ is a sequence of finite* **S4F** *tableaux such that τ_0 is an atomic tableaux and for each $n <$ sup(I), τ_{n+1} is constructed from τ_n by an application of clause* (ii)*, then $\tau = \bigcup_{n\in I} \tau_n$ is an* **S4F** *tableaux.*

Definition 4.1.5 *Let τ be an* **S4F** *tableaux.*

- *A path P in τ is* contradictory *if for some $\varphi \in \mathcal{L}_{\Box a}$ and $t \in \mathbf{W}(\mathbf{F})$, both:*

$$T[\ t \Vdash \varphi\ ] \quad and \quad F[\ t \Vdash \varphi\ ]$$

  are entries on P.

- *τ is* contradictory *if every path in τ is contradictory.*

- *τ is a* tableaux proof *of $\varphi \in \mathcal{L}_{\Box a}$ if τ has root entry $F[\ \mathbf{w}_i \Vdash \varphi\ ]$, for some $\mathbf{w}_i \in \mathbf{W}$, and τ is contradictory.*

We write

$$\mathbf{S4F} \vdash_T \varphi$$

if the formula $\varphi \in \mathcal{L}_{\Box a}$ has an **S4F** *tableaux proof.*

Note that by clause (ii) of the definition of **S4F** and **S4C** tableaux, only non-contradictory paths can be extended, thus any contradictory path is finite. Hence if τ is a contradictory tableaux, then by König's Lemma, τ is finite.

1	$F[\ \mathbf{w}_0 \Vdash [a](\varphi \to \psi) \to ([a]\varphi \to [c]\psi)\ ]\ \checkmark$			
2	$T[\ \mathbf{w}_0 \Vdash [a](\varphi \to \psi)\ ]\ \checkmark$			1:$(F \to)$
3	$F[\ \mathbf{w}_0 \Vdash [a]\varphi \to [c]\psi\ ]\ \checkmark$			1:$(F \to)$
4	$T[\ \mathbf{w}_0 \Vdash [a]\varphi\ ]\ \checkmark$			3:$(F \to)$
5	$F[\ \mathbf{w}_0 \Vdash [a]\psi\ ]\ \checkmark$			3:$(F \to)$
6	$F[\ \mathbf{F}(\mathbf{w}_0) \Vdash \psi\ ]$			5:$(F[a])$
7	$T[\ \mathbf{F}(\mathbf{w}_0) \Vdash \varphi \to \psi\ ]\ \checkmark$			2:$(T[a])$
8	$T[\ \mathbf{F}(\mathbf{w}_0) \Vdash \varphi\ ]$			4:$(T[a])$
9	$F[\ \mathbf{F}(\mathbf{w}_0) \Vdash \varphi\ ]$	7:$(T \to)$	$T[\ \mathbf{F}(\mathbf{w}_0) \Vdash \psi\ ]$	7:$(T \to)$
	$\otimes$	8,9	$\otimes$	9,6

Figure 4.1: **S4F** tableaux proof of axiom $[a]\mathbf{K}$

As stated in the (**Develop**) construction rule, the signed forcing assertion E that is being developed should formally be repeated when the corresponding atomic tableaux is appended to the path P. In our examples above, we omit this repetition as a notational convenience.

Let's start by giving tableaux proofs of the $[a]$ axioms, found in Figures 4.1 and 4.2.

1	$F[\,\mathbf{w} \Vdash \neg[a]\varphi \to [a]\neg\varphi\,]$ ✓	
2	$T[\,\mathbf{w} \Vdash \neg[a]\varphi\,]$ ✓	1:$(F \to)$
3	$F[\,\mathbf{w} \Vdash [a]\neg\varphi\,]$ ✓	1:$(F \to)$
4	$F[\,\mathbf{w} \Vdash [a]\varphi\,]$ ✓	2:$(T\neg)$
5	$F[\,\mathbf{F}(\mathbf{w}) \Vdash \neg\varphi\,]$ ✓	3:$(F[a])$
6	$F[\,\mathbf{F}(\mathbf{w}) \Vdash \varphi\,]$	4:$(F[a])$
7	$T[\,\mathbf{F}(\mathbf{w}) \Vdash \varphi\,]$	5:$(F\neg)$
	$\otimes$	7,6

1	$F[\,\mathbf{w} \Vdash [a]\neg\varphi \to \neg[a]\varphi\,]$ ✓	
2	$T[\,\mathbf{w} \Vdash [a]\neg\varphi\,]$ ✓	1:$(F \to)$
3	$F[\,\mathbf{w} \Vdash \neg[a]\varphi\,]$ ✓	1:$(F \to)$
4	$T[\,\mathbf{w} \Vdash [a]\varphi\,]$ ✓	3:$(F\neg)$
5	$T[\,\mathbf{F}(\mathbf{w}) \Vdash \neg\varphi\,]$ ✓	2:$(T[a])$
6	$F[\,\mathbf{F}(\mathbf{w}) \Vdash \varphi\,]$	5:$(T\neg)$
7	$T[\,\mathbf{F}(\mathbf{w}) \Vdash \varphi\,]$	4:$(T[a])$
	$\otimes$	7,6

Figure 4.2: **S4F** tableaux proofs of axioms $[a]\mathbf{D}$ and $[a]\mathbf{D}_c$

To see that the $[a]$-necessitation rule of the Hilbert-style proof system is preserved in $\mathbf{S4F}_T$, assume $\mathbf{S4F}_T \vdash \varphi$, and let τ be a tableaux proof with root $F[\,\mathbf{w}_0 \Vdash \varphi\,]$.

Now let $\mathbf{w}_i$ be a world symbol that does not occur in τ, and consider the atomic tableaux:

$$\begin{array}{lll} 1 & F[\,\mathbf{w}_i \Vdash [a]\varphi\,] \;\checkmark & \\ 2 & F[\,\mathbf{F}(\mathbf{w}_i) \Vdash \varphi\,] & 1{:}(F[a]) \end{array}$$

Let τ' be the result of substituting the world term $\mathbf{F}(\mathbf{w}_i)$ for $\mathbf{w}_0$ throughout τ. Note that since $\mathbf{w}_0$ is the subject of the root entry of τ, the only accessibility assertion containing $\mathbf{w}_0$ that can occur in τ is $\mathbf{w}_0\mathbf{R}\mathbf{w}_0$. Appending τ' to the atomic tableaux above will create a tableaux proof τ'' of $[a]\varphi$. Hence $\mathbf{S4F}_T \vdash [a]\varphi$.

The verification of the $\Box$-necessitation rule is similar. Assume $\mathbf{S4F}_T \vdash \varphi$, and let τ be a tableaux proof with root $F[\,\mathbf{w}_0 \Vdash \varphi\,]$. Now let $i \in \mathbb{N}$ be such that if $\mathbf{w}_j$ occurs in any entry in τ, then $j < i$. Consider the atomic tableaux:

$$\begin{array}{lll} 1 & F[\,\mathbf{w}_i \Vdash \Box\varphi\,] \;\checkmark & \\ 2 & \quad\mathbf{w}_i\mathbf{R}\mathbf{w}_{i+1} & 1{:}(F\Box) \\ 3 & F[\,\mathbf{w}_{i+1} \Vdash \varphi\,] & 1{:}(F\Box) \end{array}$$

Let τ' be the result of substituting $\mathbf{w}_{i+1}$ for $\mathbf{w}_0$ throughout τ. Appending τ' to the atomic tableaux above will create a tableaux proof τ'' of $\Box\varphi$. Hence $\mathbf{S4F}_T \vdash \Box\varphi$.

For modus ponens, the construction of a tableaux proof of ψ from tableaux proofs of φ and $\varphi \to \psi$ is essentially equivalent to giving a cut-elimination algorithm for the corresponding sequent calculus.

Now consider an **S4F** tableaux for the continuity axiom:

$$\textbf{Cont}: \quad [a]\Box\varphi \to \Box[a]\varphi$$

given in Figure 4.3, where we take φ to be any fixed atomic proposition $p \in AP$.

By inspection of the tableaux, what we need is $\mathbf{F}(\mathbf{w}_0)\,\mathbf{R}\,\mathbf{F}(\mathbf{w}_1)$, so that from (5) by $(T\Box)$, we would get $T[\,\mathbf{F}(\mathbf{w}_1) \Vdash p\,]$, and a contradiction with (12).

Definition 4.1.6 *The class of* **S4C** tableaux *is an extension of the class of* **S4F** *tableaux obtained by adding to clause* (ii) *in Definition 4.1.4 the additional construction rule:*

> (**F**-***Cont***)*: An accessibility assertion* $\mathbf{t}\mathbf{R}s$ *occurs as an entry on P,*
> *and* τ' *is constructed from* τ *by adjoining*
> *to the end of* P *the entry* $\mathbf{F}(t)\,\mathbf{R}\,\mathbf{F}(s)$*.*

We write

$$\mathbf{S4C} \vdash_T \varphi$$

if the formula $\varphi \in \mathcal{L}_{\Box a}$ *has an* **S4C** *tableaux proof, where the notion of tableaux proof for* **S4C** *is the same as that for* **S4F**.

Of course, we now trivially have an **S4C** tableaux proof of the continuity axiom.

1	$F[\ \mathbf{w}_0 \Vdash [a]\Box p \to \Box[a]p\ ]\ \checkmark$	
2	$T[\ \mathbf{w}_0 \Vdash [a]\Box p\ ]\ \checkmark$	1:$(F \to)$
3	$F[\ \mathbf{w}_0 \Vdash \Box[a]p\ ]\ \checkmark$	1:$(F \to)$
4	$\mathbf{w}_0 \mathbf{R} \mathbf{w}_0$	1:(**R-Reflex**)
5	$T[\ \mathbf{F}(\mathbf{w}_0) \Vdash \Box p\ ]\ -$	2:$(T[a])$
6	$\mathbf{F}(\mathbf{w}_0)\,\mathbf{R}\,\mathbf{F}(\mathbf{w}_0)$	1:(**R-Reflex**)
7	$\mathbf{w}_0 \mathbf{R} \mathbf{w}_1$	3:$(F\Box)$
8	$F[\ \mathbf{w}_1 \Vdash [a]p\ ]\ \checkmark$	3:$(F\Box)$
9	$\mathbf{w}_1 \mathbf{R} \mathbf{w}_1$	8:(**R-Reflex**)
10	$T[\ \mathbf{F}(\mathbf{w}_0) \Vdash p\ ]$	5,6:$(T\Box)$
11	$\mathbf{F}^2(\mathbf{w}_0)\,\mathbf{R}\,\mathbf{F}^2(\mathbf{w}_0)$	1:(**R-Reflex**)
12	$F[\ \mathbf{F}(\mathbf{w}_1) \Vdash p\ ]$	8:$(F[a])$
13	$\mathbf{F}(\mathbf{w}_1)\,\mathbf{R}\,\mathbf{F}(\mathbf{w}_1)$	8:(**R-Reflex**)
	$\vdots$	
	no contradiction	

Figure 4.3: **S4F** tableaux for axiom **Cont**

4.2 The Term Frame of a Path

The great attraction of *semantic* tableaux as a proof system is that the construction of a tableaux proof is simultaneously an attempt to build a countermodel. This section gives a careful account of the term frame of a path through a tableaux.

Definition 4.2.1 *Let τ be an* **S4F** *tableaux and let P be a path through τ.*

We associate with P a unique Kripke frame $\mathcal{K}_P = (W_P, R_P, F_P)$, called the term frame *for P, as follows.*

Let W_0 be the set of all world symbols $\mathbf{w}_i \in \mathbf{W}$ that are the subject of a signed

forcing assertion on P. Let

$$W_P = \{\mathbf{F}^k(\mathrm{w}_i) \mid \mathrm{w}_i \in W_0 \text{ and } k \in \mathbb{N}\}$$

i.e. $W_P \subseteq \mathbf{W}(\mathbf{F})$ *is the smallest subset of* $\mathbf{W}(\mathbf{F})$ *that contains all world terms that are the subject of some signed forcing assertion on P and is also closed under application of* $\mathbf{F}$. *[Note that* W_P *is always countably infinite.]*

The relation R_P *on* W_P *is defined to be the reflexive and transitive closure of the relation* R *on* W_P *defined by:*

$$(t, s) \in R \quad \Leftrightarrow \quad t\mathbf{R}s \ \text{ is an entry on } P$$

for all $t, s \in W_P$. *That is,*

$$R_P \doteq \bigcup_{m \in \mathbf{N}} R_m$$

where

$$\begin{array}{lcl} R_0 & = & \{(t,t) \mid t \in W_P\}, \text{ the identity relation on } W_P \\ R_1 & = & R = \{(t,s) \in W_P \times W_P \mid t\mathbf{R}s \ \text{ is an entry on } P\} \\ R_{m+1} & = & R_m \cup \{(t,s) \mid (\exists r \in W_P)\,(t,r) \in R_m \text{ and } (r,s) \in R_m\} \end{array}$$

Define a function $F_P : W_P \to W_P$ *by*

$$F_P(t) = \mathbf{F}(t)$$

for all $t \in W_P$*; i.e.* F_P *is the* term constructor *function on* W_P.

Definition 4.2.2 *Let* τ *be an* **S4C** *tableaux and let P be a path through* τ. *We associate with P a unique Kripke frame* $\mathcal{K}_P = (W_P, R_P, F_P)$, *called the* term frame *for P, as follows.*

The set of world terms W_P *as well as the function* $F_P : W_P \to W_P$ *is the same as in Definition 4.2.1.*

The relation R_P *on* W_P *is defined to be the reflexive, transitive and* $\mathbf{F}$*-functional closure of the relation* R *on* W_P *defined by:*

$$(t, s) \in R \quad \Leftrightarrow \quad t\mathbf{R}s \ \text{ is an entry on } P$$

for all $t, s \in W_P$. *That is,*

$$R_P \doteq \bigcup_{m \in \mathbf{N}} R_m^+$$

where

$$\begin{array}{lll} R_0^+ & = & \{(t,t) \mid t \in W_P\},\ \textit{the identity relation on } W_P \\ R_1^+ & = & R = \{(t,s) \in W_P \times W_P \mid t\mathbf{R}s \ \textit{ is an entry on } P\} \\ R_{m+1}^+ & = & R_m^+ \cup \{(t,s) \mid (\exists r \in W_P)\ (t,r) \in R_m^+ \textit{ and } (r,s) \in R_m^+\} \\ & & \cup\{(\mathbf{F}(t),\mathbf{F}(s)) \mid (t,s) \in R_m^+\} \end{array}$$

Proposition 4.2.3 *Let τ be an* **S4C** *tableaux and let P be a path through τ. Then the term frame $\mathcal{K}_P = (W_P, R_P, F_P)$ is a continuous Kripke frame.*

Proof. The R_P-monotonicity condition:

$$(t,s) \in R_P \quad \Rightarrow \quad (F_P(t), F_P(s)) \in R_P$$

for all $t, s \in W_P$, follows trivially from the definition of R_P and F_P. ■

The term frame $\mathcal{K}_P = (W_P, R_P, F_P)$ represents all the frame information expressed in entries on the path P. The signed forcing assertions occurring on P *potentially* define a valuation η_P for $\mathcal{K}_P$ satisfying:

$$\begin{array}{lll} T[\, t \Vdash \varphi\,] \text{ is an entry on P} & \Rightarrow & t \Vdash^{\mathcal{K}_P}_{\eta_P} \varphi \\ F[\, t \Vdash \varphi\,] \text{ is an entry on P} & \Rightarrow & t \nVdash^{\mathcal{K}_P}_{\eta_P} \varphi \end{array}$$

Of course, if P is a contradictory path, then there can be no valuation η_P satisfying these conditions.

Definition 4.2.4 *Let τ be an* **S4F** *(***S4C***) tableaux and suppose there is a non-contradictory path P through τ. Let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P. The* path valuation *$\eta_P : W_P \to \mathcal{P}(AP)$ for $\mathcal{K}_P$ is defined by:*

$$p \in \eta_P(t) \quad \Leftrightarrow \quad T[\, t \Vdash p\,] \textit{ is an entry on } \mathrm{P}$$

for all $t \in W_P$ and $p \in AP$.

For non-contradictory paths P, the term model $(\mathcal{K}_P,\eta_P)$ is basically a *two-sorted Herbrand structure*. Since we are working in an extension of classical propositional logic, the first sort is just the Boolean values $\mathbb{B} = \{0,1\}$, the values of $\mathcal{L}_{\Box a}$ formulas in worlds. The second sort is the real novelty: we have objects denoted by world terms $\mathbf{F}^k(\mathrm{w}_i)$ in W_P. The structure $\mathcal{K}_P$ comes equipped with a distinguished binary predicate R_P and a distinguished unary function F_P, both defined on the set W_P.

Accordingly, signed forcing assertions $S[\, t \Vdash \varphi \,]$ can be thought of as sentences in a two-sorted (first-order) language.

In making use of the term frame, one of the central notions is that of a Kripke model *agreeing with a path* P, which basically means that the model is a *quotient* of the term model $(\mathcal{K}_{\mathrm{P}},\eta_{\mathrm{P}})$.

Definition 4.2.5 *Let τ be an* **S4F** *(***S4C***) tableaux, let P be a path through τ and let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P. Let $\mathcal{K} = (W, R, F)$ be a frame for $\mathcal{L}_{\Box a}$ and let η be a valuation for $\mathcal{K}$.*

We say the Kripke model $(\mathcal{K}, \eta)$ agrees with *P iff there exists a surjective map $h : W_P \to W$ such that*

(i) *$\mathcal{K}$ is a quotient under h of $\mathcal{K}_P$, as in Definition 3.3.1, and*

(ii) *h preserves the valuations described on P:*

$$\begin{array}{lll} T[\, t \Vdash \varphi \,] \text{ is an entry on P} & \Rightarrow & h(t) \Vdash^{\mathcal{K}}_{\eta} \varphi \\ F[\, t \Vdash \varphi \,] \text{ is an entry on P} & \Rightarrow & h(t) \nVdash^{\mathcal{K}}_{\eta} \varphi \end{array}$$

for all $\varphi \in \mathcal{L}_{\Box a}$ and $t \in W_P$.

We say $(\mathcal{K}, \eta)$ agrees with P with quotient map *h, or* witnessed by *h, when we need to identify the map h.*

In the proof of completeness, we give a systematic procedure which ensures that every entry that could be on P, is on P, if the path is non-contradictory. In this case, we have $(\mathcal{K}_{\mathrm{P}},\eta_{\mathrm{P}})$ agreeing with P via the identity map. In the proof of the finite model property, we obtain a finite quotient of $(\mathcal{K}_{\mathrm{P}},\eta_{\mathrm{P}})$ agreeing with P.

The remainder of this section is devoted to identifying the key properties of the term frame $\mathcal{K}_{\mathrm{P}} = (W_{\mathrm{P}}, R_{\mathrm{P}}, F_{\mathrm{P}})$. We have already noted that W_{P} is always *countably infinite.* Note also that all the world terms generated from distinct world symbols occurring on P are distinct:

$$\mathbf{F}^k(\mathbf{w}_i) \neq \mathbf{F}^l(\mathbf{w}_j)$$

whenever $i \neq j$ or $k \neq l$. Since

$$F_{\mathrm{P}}(\mathbf{F}^k(\mathbf{w}_i)) = \mathbf{F}^{k+1}(\mathbf{w}_i)$$

the function F_{P} is trivially *injective.* World terms only get to be identified when we take a quotient of $\mathcal{K}_{\mathrm{P}}$.

Lemma 4.2.6 *Let* P *be a path through an* **S4F** *(***S4C***) tableaux* τ *, let* τ' *be an* **S4F** *(***S4C***) tableaux obtained from* τ *by extending* P *by applying one of the tableaux construction rules in clause* (ii) *of Definition 4.1.4 (Definition 4.1.6), and let* P′ *be any path through* τ' *extending* P.

(a) *If the rule applied is* not *the* $(F\Box)$ *case of the* (**Develop**) *rule, then we have:*

$$\mathcal{K}_{P'} = \mathcal{K}_P$$

(b) *If the rule* $(F\Box)$ *is applied to an entry* $F[\ t \Vdash \Box\psi\ ]$ *occurring on* P, *where* $t = \mathbf{F}^k(\mathrm{w}_i) \in W_P$, *and* $j \in \mathbb{N}$ *is the least such that* w_j *is yet to occur in any entry on* τ, *then:*

- $i < j$;
- $W_{P'} = W_P \cup \{\mathbf{F}^k(\mathrm{w}_j) \mid k \in \mathbb{N}\}$;
- $R_{P'}$ *is the reflexive and transitive closure in* $W_{P'}$ *(reflexive, transitive and* **F***-functional closure in* $W_{P'}$*) of* $R_P \cup \{(\mathbf{F}^k(\mathrm{w}_i), \mathrm{w}_j)\}$; *and*
- $F_{P'}$ *is the term constructor function on* $W_{P'}$ *uniquely extending* F_P.

Hence:

$$\mathcal{K}_P \text{ is a proper subframe of } \mathcal{K}_{P'}$$

Proof. There is only one extension P′ of P except for an application of the $(T \rightarrow)$ case of the (**Develop**) rule, when there are two possibilities for P′.

In all cases of the (**Develop**) rule other than $(F\Box)$, we have $W_{\mathrm{P}'} = W_{\mathrm{P}}$ and hence $F_{\mathrm{P}'} = F_{\mathrm{P}}$, since no new world symbols are introduced. In these cases, no new accessibility assertions are added to P so $R_{\mathrm{P}'} = R_{\mathrm{P}}$. Hence $\mathcal{K}_{\mathrm{P}'} = \mathcal{K}_{\mathrm{P}}$.

For the accessibility relation rules (**R-Reflex**) and (**R-Trans**) (and the (**F-Cont**) rule for **S4C** tableaux), there are also no new world symbols introduced, so $W_{\mathrm{P}'} = W_{\mathrm{P}}$ and $F_{\mathrm{P}'} = F_{\mathrm{P}}$. In these cases, the sole path P′ extending P only contains new accessibility assertions, and these new assertions are already accounted for in $R_{\mathrm{P}} = R_{\mathrm{P}'}$, since R_{P} is the reflexive and transitive (and **F**-functional) closure of the relation on $W_{\mathrm{P}} = W_{\mathrm{P}'}$ defined by the set of all accessibility assertions occurring on P. Hence $\mathcal{K}_{\mathrm{P}'} = \mathcal{K}_{\mathrm{P}}$ in these cases also.

It is then clear that the only case in which $\mathcal{K}_{\mathrm{P}'}$ is a proper extension of $\mathcal{K}_{\mathrm{P}}$ is in the $(F\Box)$ case. Suppose $(F\Box)$ is applied to an entry $F[\ t \Vdash \Box\psi\ ]$ on P, where $t = \mathbf{F}^k(\mathrm{w}_i)$ for some $i, k \in \mathbb{N}$, and $j \in \mathbb{N}$ is the least such that w_j is yet to occur in any entry on τ. Then we must have $i < j$, and the sole path P′ extending P contains a new accessibility assertion $\mathbf{F}^k(\mathrm{w}_i)\mathbf{R}\mathrm{w}_j$, and a new signed forcing assertion $F[\ \mathrm{w}_j \Vdash \psi\ ]$.

By the definition of the term frame of a path, the extension $\mathcal{K}_{P'}$ clearly satisfies (b). ■

So accessibility assertions added to a path in an **S4F** or **S4C** tableaux by $(F\Box)$ are always of the form:

$$\mathbf{F}^k(\mathrm{w}_i)\,\mathbf{R}\,\mathrm{w}_j \quad \text{where} \quad i < j$$

The pattern is always *from* an arbitrary (complex or simple) world term *to* a later *simple* world term. For **S4F**, taking the reflexive and transitive closure does not change this fundamental pattern, while for **S4C**, taking the **F**-functional closure opens up more possibilities but the pattern is still quite constrained.

Proposition 4.2.7 *Let τ be an* **S4F** *tableaux, let P be a path through τ, and let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P.*

Then for each $t = \mathbf{F}^k(\mathrm{w}_i) \in W_P$, every R_P chain from t is of the form:

$$\langle \mathbf{F}^k(\mathrm{w}_i)\rangle * \langle \mathrm{w}_{i_j} \mid j \in J\rangle$$

for some $\emptyset \preceq J \preceq \mathbb{N}$, where

$$i < i_0 < i_j < i_{j+1}$$

for all $j < \sup(J)$.

So the term frame $\mathcal{K}_P$ has the following properties: W_P is countably infinite, R_P is a partial order, and F_P is injective.

Hence the induced D-topological structure $\mathfrak{T}_P = (W_P, \mathcal{T}_P, F_P)$ is countable and T_0, with an injective function.

Moreover, the basic open set for $t = \mathbf{F}^k(\mathrm{w}_i) \in W_P$ in the cone topology $\mathcal{T}_P = \mathcal{T}_{R_P}$ is of the form

$$B_t = \{\mathbf{F}^k(\mathrm{w}_i)\} \cup \{\mathrm{w}_{i_n} \mid n \in N\}$$

for some (possibly empty) subset $N \subseteq \mathbb{N}$, where $i < i_n < i_{n'}$ for all $n, n' \in N$ with $n < n'$.

Proof. Fix $t = \mathbf{F}^k(\mathrm{w}_i) \in W_P$. It is clear from the construction of **S4F** tableaux that for all $s \in \mathbf{W}(\mathbf{F})$, if $t\mathbf{R}s$ is an entry on P, then *either* $s = t$, *or else* $s = \mathrm{w}_l \in \mathbf{W}$ for some $l > i$ and there exists $m \geq 0$ and world symbols $\mathrm{w}_{i_0}, \mathrm{w}_{i_1}, \ldots, \mathrm{w}_{i_m} \in \mathbf{W}$ such that

- $i < i_0 < i_1 < \ldots < i_m = l$;
- the world symbol w_{i_0} was introduced by $(F\Box)$ applied to a signed forcing assertion $F[\, t \Vdash \Box\psi_1 \,]$ on P, and $t\mathbf{R}\mathrm{w}_{i_0}$ is an entry on P;

- for $1 \leq j < m$, the world symbol $\mathbf{w}_{i_{j+1}}$ was introduced by $(F\Box)$ applied to a signed forcing assertion $F[\ \mathbf{w}_{i_j} \Vdash \Box\psi_j\ ]$ on P, and $w_{i_j}\mathbf{R}\mathbf{w}_{i_{j+1}}$ is an entry on P.

Recall from Definition 4.2.1 that

$$R_{\mathrm{P}} = \bigcup_{n \in \mathrm{N}} R_n$$

It then follows by induction on n that if $(t, s) \in R_n$ and $s \neq t$, then $s = \mathbf{w}_l \in \mathbf{W}$ for some $l > i$ and there exists $m \geq 0$ and world symbols $\mathbf{w}_{i_0}, \mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_m} \in \mathbf{W}$ such that $i < i_0 < i_1 < \ldots < i_m = l$; $(t, \mathbf{w}_{i_0}) \in R_n$; and for $1 \leq j < m$, $(\mathbf{w}_{i_j}, \mathbf{w}_{i_{j+1}}) \in R_n$.

Hence every R_{P} chain from t is of the form:

$$\langle \mathbf{F}^k(\mathbf{w}_i) \rangle * \langle \mathbf{w}_{i_j} \mid j \in J \rangle$$

for some $\emptyset \preceq J \preceq \mathbb{N}$, where

$$i < i_0 < i_j < i_{j+1}$$

for all $j < \sup(J)$.

Now suppose $(t, s) \in R_{\mathrm{P}}$ with $s \neq t$. Then $s = \mathbf{w}_j$ for some $j > i$, so $(s, t) = (\mathbf{w}_j, \mathbf{F}^k(\mathbf{w}_i)) \notin R_{\mathrm{P}}$. Hence R_{P} is a partial order, as required. The remaining properties of $\mathcal{K}_{\mathrm{P}}$ have already been noted.

Finally, observe that in the cone topology $\mathcal{T}_{\mathrm{P}}$, the basic open set B_t is the union of all R_{P} chains from t. ■

When P is a path through an **S4C** tableaux, the R_{P} chains are more complicated but still admit an explicit description.

Proposition 4.2.8 *Let τ be an* **S4C** *tableaux, let P be a path through τ, and let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P.*

Then for each $t = \mathbf{F}^k(\mathbf{w}_i) \in W_P$, every R_P chain from t is of the form:

$$\langle \mathbf{F}^k(\mathbf{w}_i) \rangle * \langle \mathbf{F}^{k_j}(\mathbf{w}_{i_j}) \mid j \in J \rangle$$

for some $\emptyset \preceq J \preceq \mathbb{N}$, where

$$i < i_0 < i_j < i_{j+1} \quad \textit{and} \quad k_{j+1} \leq k_j \leq k_0 \leq k$$

for all $j < \sup(J)$.

So the term frame $\mathcal{K}_P$ has the following properties: W_P is countably infinite, R_P is a partial order, and F_P is continuous and injective.

Hence the induced continuous D-topological structure $\mathfrak{T}_P = (W_P, \mathcal{T}_P, F_P)$ is countable and T_0, with an injective function.

Moreover, the basic open set for $t = \mathbf{F}^k(\mathbf{w}_i) \in W_{\mathsf{P}}$ *in the cone topology* $\mathcal{T}_{\mathsf{P}} = \mathcal{T}_{R_{\mathsf{P}}}$ *is of the form*

$$B_t = \{\mathbf{F}^k(\mathbf{w}_i)\} \cup \{\mathbf{F}^{k_s}(\mathbf{w}_{i_n}) \mid n \in N\}$$

for some (possibly empty) subset $N \subseteq \mathbb{N}$*, where* $i < i_n < i_{n'}$ *and* $k_n \leq k$ *for all* $n, n' \in N$ *with* $n < n'$*.*

Proof. Fix $t = \mathbf{F}^k(\mathbf{w}_i) \in W_{\mathsf{P}}$. It is clear from the definition of **S4C** tableaux that for all $s \in \mathbf{W}(\mathbf{F})$, if $t\mathbf{R}s$ is an entry on P, then *either* $s = t$, *or else* there exists $m \geq 0$, world symbols $\mathbf{w}_{i_0}, \mathbf{w}_{i_1}, ..., \mathbf{w}_{i_m} \in \mathbf{W}$ and world terms $t_0, t_1, ..., t_m \in \mathbf{W}(\mathbf{F})$ such that

- for $0 \leq j \leq m$, $t_j = \mathbf{F}^{k_j}(\mathbf{w}_{i_j})$ for some $k_j \geq 0$;
- for $0 \leq j < m$, $t_j\mathbf{R}t_{j+1}$ is an entry on P;
- $t_m = s$;
- $i < i_0 < i_1 < ... < i_m$;
- the world symbol $\mathbf{w}_{i_0}$ was introduced by $(F\Box)$ applied to a signed forcing assertion $F[\,\mathbf{F}^l(\mathbf{w}_i) \Vdash \Box\psi_0\,]$ on P, for some $l \geq 0$, and $\mathbf{F}^l(\mathbf{w}_i)\mathbf{R}\mathbf{w}_{i_0}$ is an entry on P;
- for $0 < h \leq k_0$, the entry $\mathbf{F}^{l+h}(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^h(\mathbf{w}_{i_0})$ is on P in virtue of the **(F-Cont)** rule applied to the entry $\mathbf{F}^{l+h-1}(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^{h-1}(\mathbf{w}_{i_0})$ on P;
- $k = l + k_0$, and so $k_0 \leq k$;
- for $0 \leq j < m$, the world symbol $\mathbf{w}_{i_{j+1}}$ was introduced by $(F\Box)$ applied to a signed forcing assertion $F[\,\mathbf{F}^{l_j}(\mathbf{w}_{i_j}) \Vdash \Box\psi_j\,]$ on P, for some $l_j \geq 0$, and $\mathbf{F}^{l_j}(\mathbf{w}_{i_j})\,\mathbf{R}\,\mathbf{w}_{i_{j+1}}$ is an entry on P;
- for $0 \leq j < m$ and for $0 < h \leq k_{j+1}$, the entry $\mathbf{F}^{l_j+h}(\mathbf{w}_{i_j})\,\mathbf{R}\,\mathbf{F}^h(\mathbf{w}_{i_{j+1}})$ is on P in virtue of the **(F-Cont)** rule applied to the entry $\mathbf{F}^{l_j+h-1}(\mathbf{w}_{i_j})\,\mathbf{R}\,\mathbf{F}^{h-1}(\mathbf{w}_{i_{j+1}})$ on P; and
- for $0 \leq j < m$, $k_j = l_j + k_{j+1}$, and so $k_{j+1} \leq k_j$.

Recall from Definition 4.2.2 that

$$R_{\mathsf{P}} = \bigcup_{n \in \mathbb{N}} R_n^+$$

It then follows by induction on n that if $(t, s) \in R_n^+$ and $s \neq t$, then there exists $m \geq 0$, world symbols $\mathbf{w}_{i_0}, \mathbf{w}_{i_1}, ..., \mathbf{w}_{i_m} \in \mathbf{W}$, and world terms $t_0, t_1, ..., t_m \in \mathbf{W}(\mathbf{F})$ such that

$i < i_0 < i_1 < \ldots < i_m = l$; for $0 \leq j \leq m$, $t_j = \mathbf{F}^{k_j}(\mathbf{w}_{i_j})$ and $(t_j, t_{j+1}) \in R_n^+$, if $j < m$; for $0 < h \leq k_0$, $(\mathbf{F}^{l+h}(\mathbf{w}_i), \mathbf{F}^l(\mathbf{w}_{i_0})) \in R_n^+$ where $k = l + k_0$; and for $1 \leq j < m$ and $0 < h \leq k_{j+1}$, $(\mathbf{F}^{l_j+h}(\mathbf{w}_{i_j}), \mathbf{F}^h(\mathbf{w}_{i_{j+1}})) \in R_n^+$ with $k_j = l_j + k_{j+1}$.

Hence every R_{P} chain from t is of the form:

$$\langle \mathbf{F}^k(\mathbf{w}_i) \rangle * \langle \mathbf{F}^{k_j}(\mathbf{w}_{i_j}) \mid j \in J \rangle$$

for some $\emptyset \preceq J \preceq \mathbb{N}$, where:

$$i < i_0 < i_j < i_{j+1} \quad \text{and} \quad k_{j+1} \leq k_j \leq k_0 \leq k$$

for all $j < \sup(J)$.

Now suppose $(t, s) \in R_{\mathrm{P}}$ with $s \neq t$. Then $s = \mathbf{F}^l(\mathbf{w}_j)$ for some $j > i$ and $l \leq k$, so $(s, t) = (\mathbf{F}^l(\mathbf{w}_j), \mathbf{F}^k(\mathbf{w}_i)) \notin R_{\mathrm{P}}$. Hence R_{P} is a partial order, and so the cone topology $\mathcal{T}_{\mathrm{P}} = \mathcal{T}_{R_{\mathrm{P}}}$ is T_0. As before, the description of the basic open set B_t follows from the fact that B_t is the union of all R_{P} chains from t. ■

Note that the continuous term frame will also satisfy the converse of the R_{P}-monotonicity condition

$$(\mathbf{F}(t), \mathbf{F}(s)) \in R_{\mathrm{P}} \quad \Rightarrow \quad (t, s) \in R_{\mathrm{P}}$$

for all $t, s \in W_{\mathrm{P}}$, since if a complex term $\mathbf{F}(s)$ occurs on the right hand side of an accessibility assertion $\mathbf{F}(t)\,\mathbf{R}\,\mathbf{F}(s)$ on P, it got put there by either the (**F-Cont**) rule or else the (**R-Reflex**) or (**R-Trans**) rules. However, as noted in our discussion at the end of Section 3.1, F_{P} will fail to be an *open map* w.r.t. the topology $\mathcal{T}_{R_{\mathrm{P}}}$ because the primitive world symbols $\mathbf{w}_i$ in W_{P} fall outside the range of F_{P}.

As a set, $W_{\mathrm{P}} \cong I \times \mathbb{N}$ for some initial segment $I \preceq \mathbb{N}$, so one can think of the term frame $\mathcal{K}_{\mathrm{P}}$ as a two-dimensional discrete array of points, vertically infinite with the points in a column with base $\mathbf{w}_i$ labelled by all the iterates $\mathbf{F}^k(\mathbf{w}_i)$ for $k \in \mathbb{N}$, but possibly finite horizontally. Accessibility relations are edges always going from left to right, from an $\mathbf{F}^k(\mathbf{w}_i)$ to a $\mathbf{w}_j$ for $j > i$, and when $\mathbf{F}$ denotes a continuous function, there are all the "parallel" edges from $\mathbf{F}^{k+l}(\mathbf{w}_i)$ to a $\mathbf{F}^l(\mathbf{w}_j)$ as well. In practice, there is a bound on the number of iterates k one needs to take. If the $[a]$-rank of a formula φ is the number of subformulas of φ of the form $[a]\psi$, then the iterates $\mathbf{F}^k(\mathbf{w}_i)$ for $0 \leq k \leq [a]-\mathrm{rank}(\varphi)$ will be sufficient. Taking all the iterates is a technical convenience to ensure that the term constructor function $t \mapsto \mathbf{F}(t)$ is total.

4.3 Soundness of Tableaux

The Soundness of **S4F** and **S4C** tableaux is a simple consequence of the following theorem (an extension of [NS97], Theorem 4.2).

Theorem 4.3.1 *Let τ be an* **S4F** *(***S4C***) tableaux, let $\mathcal{K} = (W, R, F)$ be a (continuous) frame for $\mathcal{L}_{\Box a}$, let η be a valuation for $\mathcal{K}$, and let $\varphi \in \mathcal{L}_{\Box a}$.*

If for some world symbol $\mathbf{w} \in \mathbf{W}$ and some element $w \in W$, either

(a) $F[\, \mathbf{w} \Vdash \varphi \,]$ *is the root entry of τ, and $w \nVdash^{\mathcal{K}}_{\eta} \varphi$, or else*

(b) $T[\, \mathbf{w} \Vdash \varphi \,]$ *is the root entry of τ, and $w \Vdash^{\mathcal{K}}_{\eta} \varphi$,*

then there is a path P through τ such that $(\mathcal{K}, \eta)$ agrees with P, where the quotient map $h : W_{\mathsf{P}} \to W$ satisfies $h(\mathbf{w}) = w$.

To prove Theorem 4.3.1, we need a further lemma on the inductive construction of tableaux.

Lemma 4.3.2 *Let τ be an* **S4F** *(***S4C***) tableaux, let P be a path through τ, let $\mathcal{K}_{\mathsf{P}} = (W_{\mathsf{P}}, R_{\mathsf{P}}, F_{\mathsf{P}})$ be the term frame of P, let $\mathcal{K} = (W, R, F)$ be a (continuous) frame for $\mathcal{L}_{\Box a}$ and let η be a valuation for $\mathcal{K}$.*

If $(\mathcal{K}, \eta)$ agrees with P, with quotient map $h : W_{\mathsf{P}} \to W$, and τ' is an **S4F** *(***S4C***) tableaux obtained from τ by extending P using one of the tableaux construction rules in clause* (ii) *of Definition 4.1.4 (Definition 4.1.6),*

then there is a path P' through τ' extending P and a function $h' : W_{\mathsf{P}'} \to W$ extending h such that h' witnesses that $(\mathcal{K}, \eta)$ agrees with P'.

Proof. By Lemma 4.2.6, if P' is any extension of P in virtue of clause (ii) of Definition 4.1.4 (Definition 4.1.6), then $W_{\mathsf{P}'} = W_{\mathsf{P}}$ and $\mathcal{K}_{\mathsf{P}} = \mathcal{K}_{\mathsf{P}'}$ *except* if P is extended to P' using the $(F\Box)$ case of the (**Develop**) rule. So for the tableaux construction rules other than $(F\Box)$, the quotient map h requires no extension and we only need show that $h = h'$ preserves the valuations described on a path P' through τ' extending P.

The propositional cases of (**Develop**) rule are straightforward: from the hypothesis of the lemma and the corresponding clause in the definition of forcing (Definition 2.3.2), it is clear that one of the (at most two) extensions P' of P will be such that h preserves the valuations described on P'.

Applications of the (**R-Reflex**) or (**R-Trans**) rules, or the (**F-Cont**) rule for **S4C** tableaux introduce no new signed forcing assertions, so it is immediate from the hypothesis of the lemma that h preserves the valuations described on P'.

When the $(T\Box)$ case of **(Develop)** rule is applied, entries of the form $T[\, t \Vdash \Box\varphi \,]$ and $t\mathbf{R}s$ occur on P, and the sole path P′ extending P contains a single new signed forcing assertion $T[\, s \Vdash \varphi \,]$. By the hypothesis and the definition of agreement for P, we have $h(t) \Vdash^{\mathcal{K}}_{\eta} \Box\varphi$ and $(h(t), h(s)) \in R$, hence by the $\Box$ clause in the definition of forcing, $h(s) \Vdash^{\mathcal{K}}_{\eta} \varphi$, so h preserves the valuations described on P′.

When the $(T[a])$ (respectively, $(F[a])$) case of **(Develop)** rule is applied, an entry of the form $T[\, t \Vdash [a]\varphi \,]$ $(F[\, t \Vdash [a]\varphi \,])$ occurs on P, and the sole path P′ extending P contains a single new signed forcing assertion $T[\, \mathbf{F}(t) \Vdash \varphi \,]$ $(F[\, \mathbf{F}(t) \Vdash \varphi \,])$. By the hypothesis and the definition of agreement for P, we have $h(t) \Vdash^{\mathcal{K}}_{\eta} [a]\varphi$ $(h(t) \nVdash^{\mathcal{K}}_{\eta} [a]\varphi)$. By the $[a]$ clause in the definition of forcing, and the commutativity equation $F(h(t)) = h(\mathbf{F}(t))$,

$$h(t) \Vdash^{\mathcal{K}}_{\eta} [a]\varphi \quad \text{iff} \quad F(h(t)) \Vdash^{\mathcal{K}}_{\eta} \varphi \quad \text{iff} \quad h(\mathbf{F}(t)) \Vdash^{\mathcal{K}}_{\eta} \varphi$$

so h preserves the valuations described on P′.

It remains to deal with the $(F\Box)$ case of the **(Develop)** rule. In this case, for some world term $t \in W_{\mathrm{P}}$, there is an entry $F[\, t \Vdash \Box\varphi \,]$ occurring on P, and the sole path P′ extending P contains a new accessibility assertion of the form $t\mathbf{R}\mathbf{w}_j$, with $\mathbf{w}_j$ a new world symbol, as well as a new signed forcing assertion $F[\, \mathbf{w}_j \Vdash \varphi \,]$. Hence

$$W_{\mathrm{P}'} = W_{\mathrm{P}} \cup \{\mathbf{F}^k(\mathbf{w}_j) \mid k \in \mathbb{N}\}$$

and $R_{\mathrm{P}'}$ is the reflexive and transitive closure (reflexive, transitive and $\mathbf{F}$-functional closure) of $R_{\mathrm{P}} \cup \{(t, \mathbf{w}_j)\}$ in $W_{\mathrm{P}'}$. Now by the hypothesis and the definition of agreement for P, we have $h(t) \nVdash^{\mathcal{K}}_{\eta} \Box\varphi$. Hence by the $\Box$ clause in the definition of forcing, there is a $w \in W$ such that $(h(t), w) \in R$ and $w \nVdash^{\mathcal{K}}_{\eta} \varphi$. Define $h' : W_{\mathrm{P}'} \to W$ by

$$h'(s) = \begin{cases} h(s) & \text{if } s \in W_{\mathrm{P}} \\ F^k(w) & \text{if } s = \mathbf{F}^k(\mathbf{w}_j) \text{ and } k \in \mathbb{N} \end{cases}$$

Then h' is surjective, and it follows that $(s, r) \in R_{\mathrm{P}'}$ implies $(h'(s), h'(r)) \in R$, and $F(h'(s)) = h'(\mathbf{F}(s))$, for all $s, r \in W_{\mathrm{P}'}$. Since $w = h'(\mathbf{w}_j)$, we have $h'(\mathbf{w}_j) \nVdash^{\mathcal{K}}_{\eta} \varphi$. Hence $(\mathcal{K}, \eta)$ agrees with P′ with witness h', as required. ■

We now return to the proof of Theorem 4.3.1.

Proof. Let τ be an **S4F** (**S4C**) tableaux, let $\mathcal{K} = (W, R, F)$ be a frame for $\mathcal{L}_{\Box a}$, let η be a valuation for $\mathcal{K}$, and let $\varphi \in \mathcal{L}_{\Box a}$. Assume that for some world symbol $\mathbf{w} \in \mathbf{W}$ and some element $w \in W$, either

(a) $F[\, \mathbf{w} \Vdash \varphi \,]$ is the root entry of τ, and $w \nVdash^{\mathcal{K}}_{\eta} \varphi$, or else

(b) $T[\, \mathbf{w} \Vdash \varphi \,]$ is the root entry of τ, and $w \Vdash^{\mathcal{K}}_{\eta} \varphi$,

We proceed by induction on the construction of the tableaux τ.

If τ is an atomic **S4F** (**S4C**) tableaux, as in clause (i) of Definition 4.1.4, then let P_r to be the path through τ consisting of the root node only. Then

$$W_{\mathsf{P}_r} = \{\mathbf{F}^k(\mathbf{w}) \mid k \in \mathbb{N}\}$$

R_{P_r} is the empty binary relation on W_{P_r}, and $F_{\mathsf{P}_r}(\mathbf{F}^k(\mathbf{w})) = \mathbf{F}^{k+1}(\mathbf{w})$ for all $k \in \mathbb{N}$. Define $h_r : W_{\mathsf{P}_r} \to W$ by

$$h_r(\mathbf{F}^k(\mathbf{w})) = F^k(w)$$

for all $k \in \mathbb{N}$. Then by the definition of h_r and the hypothesis of the theorem, h_r witnesses that $(\mathcal{K}, \eta)$ agrees with P_r and $h_r(\mathbf{w}) = w$. Now apply the same reasoning as in the proof of Lemma 4.3.2 to obtain a path P through τ extending P_r and a function $h : W_{\mathsf{P}} \to W$ extending h_r such that h witnesses that $(\mathcal{K}, \eta)$ agrees with P and $h(\mathbf{w}) = w$.

Next, suppose τ is an **S4F** (**S4C**) tableaux and $(\mathcal{K}, \eta)$ agrees with a path P through τ, with quotient map $h : W_{\mathsf{P}} \to W$ such that $h(\mathbf{w}) = w$, and τ' is an **S4F** (**S4C**) tableaux obtained from τ by extending P using one of the tableaux construction rules in clause (ii) of Definition 4.1.4 (Definition 4.1.6). Then by Lemma 4.3.2, there is a path P' through τ' extending P and a function $h' : W_{\mathsf{P}'} \to W$ extending h such that h' witnesses that $(\mathcal{K}, \eta)$ agrees with P' and $h'(\mathbf{w}) = w$.

Finally, suppose that $I \preceq \mathbb{N}$ and $\tau = \bigcup_{n \in I} \tau_n$ is an **S4F** (**S4C**) tableaux as in clause (iii) of Definition 4.1.4 (Definition 4.1.6), where τ_0 is an atomic tableaux and for each $n < \sup(I)$, τ_{n+1} is constructed from τ_n by an application of clause (ii) of that definition. Apply the argument as in the atomic case above to obtain a path P_0 through τ_0 and a function $h_0 : W_{\mathsf{P}_0} \to W$ such that h_0 witnesses that $(\mathcal{K}, \eta)$ agrees with P_0 and $h_0(\mathbf{w}) = w$. Then for each $n \in I$, apply Lemma 4.3.2 to the path P_n through τ_n and function $h_n : W_{\mathsf{P}_n} \to W$ witnessing that $(\mathcal{K}, \eta)$ agrees with P_n with $h_n(\mathbf{w}) = w$, to obtain a path P_{n+1} through τ_{n+1} extending P_n and a function $h_{n+1} : W_{\mathsf{P}_{n+1}} \to W$ extending h_n, witnessing that $(\mathcal{K}, \eta)$ agrees with P_{n+1} and satisfying $h_{n+1}(\mathbf{w}) = w$. Then set $\mathsf{P} = \bigcup_{n \in I} \mathsf{P}_n$ and $h = \bigcup_{n \in I} h_n$ to obtain a path P through τ such that $(\mathcal{K}, \eta)$ agrees with P, with the quotient map $h : W_{\mathsf{P}} \to W$ satisfying $h(\mathbf{w}) = w$. ■

Theorem 4.3.3 Kripke Soundness of **S4F** and **S4C** tableaux
For all formulas φ of $\mathcal{L}_{\Box a}$,
if **S4F** $\vdash_T \varphi$ *(***S4C** $\vdash_T \varphi$*) then for all Kripke frames $\mathcal{K}$ for $\mathcal{L}_{\Box a}$, $\mathcal{K} \Vdash \varphi$.*

Proof. Suppose τ is an **S4F** (**S4C**) tableaux proof of φ, with root $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$, and suppose for a contradiction that φ is not Kripke valid. Then there is a frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$, a valuation η for $\mathcal{K}$ and a world $w_0 \in W$ such that $w_0 \nVdash_{\eta}^{\mathcal{K}} \varphi$.

By Theorem 4.3.1, there is a path P through τ such that $(\mathcal{K}, \eta)$ agrees with P, with the quotient map $h : W_{\mathrm{P}} \to W$ satisfying $h(\mathbf{w}_0) = w_0$. Since τ is a tableaux proof, P is contradictory, so there is a world term $t \in W_{\mathrm{P}}$ and a subformula ψ of φ such that both $T[\ t \Vdash \psi\ ]$ and $F[\ t \Vdash \psi\ ]$ occur as entries on P. Since $(\mathcal{K}, \eta)$ agrees with P with quotient map h, we have as a contradiction both $v \Vdash^{\mathcal{K}}_{\eta} \psi$ and $v \nVdash^{\mathcal{K}}_{\eta} \psi$, where $v = h(t) \in W$. ■

4.4 Completeness of Tableaux

The construction of a tableaux is a non-deterministic procedure. To prove completeness, we give a *systematic* procedure for developing a tableaux, so that every entry that *could* occur on a non-contradictory path, *does*. We define the *complete systematic tableaux* starting with a given signed forcing assertion as its root entry. We then prove that for any non-contradictory path P through this tableaux, $(\mathcal{K}_{\mathrm{P}}, \eta_{\mathrm{P}})$ agrees with P under the identity map. Thus if this systematic development of a tableaux τ with root entry $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$ fails to produce a tableaux proof of φ, then we obtain a Kripke counter-model for φ, and so demonstrate that φ is not Kripke valid.

In this section on completeness, we are concerned with *model-existence*, via the existence of a non-contradictory path through the complete systematic tableaux with root entry $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$, for each formula φ of $\mathcal{L}_{\Box a}$ that is not **S4F** or **S4C** tableaux provable. In general, such non-contradictory paths are infinite, and as we have defined it, the term frame of any path is infinite because we take *all* iterates $\mathbf{F}^k(\mathbf{w}_i)$ of each world symbol $\mathbf{w}_i$ that is the subject of a signed forcing assertion on the path. In Section 4.5 below, we establish the finite model property by specifying a suitable quotient of the term frame.

Recall from Definition 4.1.4 that when a signed forcing assertion E is being developed on a path P, we formally require that the entry E be repeated when the corresponding atomic tableaux with root entry E is appended to P. Most signed forcing assertions occur at most twice on a path: once (called occurrence 0) when the entry first appears, either as the root entry or by one of the cases of the (**Develop**) rule, and again (occurrence 1) if it gets to be developed by any case *other than* the $(T\Box)$ case of the (**Develop**) rule. In the $(T\Box)$ case, an entry $E = T[\ t \Vdash \Box\psi\ ]$ may be developed and hence repeated infinitely often, *at least* once for *each* world term $s_j \in \mathbf{W}(\mathbf{F})$. For the purposes of defining complete systematic tableaux, we make a cosmetic modification to the $(T\Box)$ case in Definition 4.1.4 so that each time an entry $E = T[\ t \Vdash \Box\psi\ ]$ is developed, a candidate world term $s_j \in \mathbf{W}(\mathbf{F})$ must be "declared". The revised version of the rule now reads:

$(T\Box)(s_j)$ case of (**Develop**): A signed forcing assertion $T[\ t \Vdash \Box\psi\ ]$ occurs on a (non-contradictory) path P in τ, and τ' is constructed from τ by appending to the end of the path P either:

(a) the atomic tableaux

$$\begin{array}{c} T[\ t \Vdash \Box\psi\ ] \\ | \\ T[\ s_j \Vdash \psi\ ] \end{array}$$

if $t\mathbf{R}s_j$ occurs on P and $T[\ s_j \Vdash \psi\ ]$ does not occur on P, or else:

(b) the atomic tableaux consisting of the sole entry

$$T[\ t \Vdash \Box\psi\ ]$$

otherwise.

So in an application of $(T\Box)(s_j)$, the entry $T[\ t \Vdash \Box\psi\ ]$ will be simply repeated, without further extension of the path P, if either $t\mathbf{R}s_j$ does not occur on P or if the s_j-instance $T[\ s_j \Vdash \psi\ ]$ already occurs on P. The transformations between tableaux constructed using the original version of the $(T\Box)$ case of the (**Develop**) rule, and tableaux constructed using the new $(T\Box)(s_j)$ cases of (**Develop**), are straightforward (albeit tedious).

We also use the pairing function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ to keep track of attempts to develop $T\Box$ entries. When we are working on occurrence m of $E = T[\ t \Vdash \Box\psi\ ]$, where $m = p(j, l)$, we make the "l^{th}" attempt at developing E using $(T\Box)(s_j)$. If either $t\mathbf{R}s_j$ does not *yet* occur on the path, or if the s_j-instance $T[\ s_j \Vdash \psi\ ]$ already occurs on the path, then the development of this occurrence m of E stops with the simple repetition of E; if $t\mathbf{R}s_j$ does occur on the path but $T[\ s_j \Vdash \psi\ ]$ does not, the path is extended with a repetition of E plus the new entry $T[\ s_j \Vdash \psi\ ]$. The accessibility assertion $t\mathbf{R}s_j$ *may* appear later, on an extension of the current path constructed using the rules (**R-Reflex**), (**R-Trans**) or (**F-Cont**) or the $(F\Box)$ case of the (**Develop**) rule if $s_j = \mathbf{w}_i$ is a world symbol that hasn't yet appeared, so we will need to make a further "$(l+1)^{\text{st}}$" attempt to develop E using $(T\Box)(s_j)$.

Definition 4.4.1 *Let $\tau = \bigcup_{n \in I} \tau_n$ be an* **S4F** *(***S4C***) tableaux and P a path through τ. Let E be a signed forcing assertion on τ and let e be occurrence m of E on P, i.e. the "m^{th}" node on* P *labelled with E, where $m = p(j, l)$, for $j, l, m \in \mathbb{N}$.*

We say e is reduced *on* P *iff one of the following cases hold:*

(i) *E is* not *of the form $T[\ t \Vdash \Box\psi\ ]$, and for some n, τ_{n+1} is obtained from τ_n by an application of the appropriate case of the (***Develop***) rule to the entry E, the tableaux τ_n and the path P_n through τ_n, where $P_n = P \restriction \tau_n$.*

(ii) *E is of the form $T[\ t \Vdash \Box\psi\ ]$, there is an occurrence $m' = p(j, l+1)$ of E on* P*, and either $t\mathbf{R}s_j$ is* not *an entry on* P *or $T[\ s_j \Vdash \psi\ ]$ is an entry on* P.

We say τ is finished *iff for every non-contradictory path* P *through τ, every occurrence of every signed forcing assertion on* P *is reduced on* P.

Lemma 4.4.2 *If*

- *τ is an* **S4F** *(***S4C***) tableaux;*
- P *a non-contradictory path in τ;*
- *τ' is an* **S4F** *(***S4C***) tableaux obtained from τ by extending* P *to non-contradictory path* P′*;*
- *e is occurrence $m = p(j, l)$ of a signed forcing assertion E on* P*; and*
- *e is reduced on* P*;*

then the only way e could fail *to be reduced on* P′ *is if*

(0) *for some $t \in \mathbf{W}(\mathbf{F})$ and $\psi \in \mathcal{L}_{\Box a}$, E is $T[\ t \Vdash \Box\psi\ ]$, and*

(1) *$t\mathbf{R}s_j$ is an entry on* P′ *but not on* P*, and*

(2) *$T[\ s_j \Vdash \psi\ ]$ is not an entry on* P′ *(and hence not an entry on* P*).*

Proof. Immediate from Definition 4.4.1. Note that since e is reduced on P, by hypothesis, then if E is $T[\ t \Vdash \Box\psi\ ]$, there is an occurrence $m' = p(j, l+1)$ of E on P and hence on any extension P′. In the process of extending P to P′, the new accessibility assertion $t\mathbf{R}s_j$ must have been appended using either the (**R-Reflex**), (**R-Trans**) or (**F-Cont**) rules, or using the ($F\Box$) case of the (**Develop**) rule if $s_j = \mathbf{w}_i$ for some world symbol $\mathbf{w}_i$ that had not yet appeared on the tableaux. ■

As a binary tree, a tableaux τ has a natural left-right ordering on the nodes (occurrences of entries) at each of its levels. As in [NS97], Definition II.6.8, define the *level-lexicographic ordering* $<_{LL}$ on nodes e, e' of a tableaux τ as follows:

$e <_{LL} e'$ iff the level of e is less that e', or else e and e' are on the same level, and e is to the left of e'

It is immediate that $<_{LL}$ is a well-ordering of the nodes e of a tableaux τ: there are only finitely many nodes of τ that are $<_{LL}$ any given node e.

Definition 4.4.3 *For each formula* φ *of* $\mathcal{L}_{\Box a}$, *we define the* complete systematic **S4F** *(***S4C***)* tableaux, **S4F**-*CST (***S4C**-*CST)* τ^{φ} *for* φ *to be:*

$$\tau^{\varphi} = \bigcup_{n \in I} \tau_n$$

for some $I \preceq N$, *where the sequence of finite* **S4F** *(***S4C***) tableaux* $\{\tau_n\}_{n\in I}$ *for* φ *is defined inductively as follows.*

τ_0 *is the atomic tableaux with root entry* $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$ *(or* $T[\ \mathbf{w}_0 \Vdash \varphi\ ]$ *). This atomic tableaux is uniquely specified by requiring that in case* $(F\Box)$, *where* E *is* $F[\ \mathbf{w}_0 \Vdash \Box\psi\ ]$, *we use the entries* $\mathbf{w}_0\mathbf{R}\mathbf{w}_1$ *and* $F[\ \mathbf{w}_1 \Vdash \psi\ ]$ *(or in the* $(T\Box)$ *case, the tableaux consists of the root entry only).*

At stage n, we have by induction a finite **S4F** *(***S4C***) tableaux* τ_n.

If τ_n *is finished, then we terminate the construction. Otherwise, we extend* τ_n *to a finite* **S4F** *(***S4C***) tableaux* τ_{n+1} *as follows.*

Case 1: $n = 4k + 1$, *for* $k \in \mathbb{N}$.

Then τ_{n+1} *is the tableaux obtained from* τ_n *by appending, to the end of each non-contradictory path P through* τ_n, *the entry* $s_j\mathbf{R}s_j$ *for the least* $j \leq k$ *such that:*

- $s_j = \mathbf{F}^k(\mathbf{w}_i)$ *and* $\mathbf{w}_i$ *is the subject of some signed forcing assertion on P, and*
- $s_j\mathbf{R}s_j$ *does not yet occur on P.*

Case 2: $n = 4k + 2$, *for* $k \in \mathbb{N}$.

Then τ_{n+1} *is the tableaux obtained from* τ_n *by appending, to the end of each non-contradictory path P through* τ_n, *the entry* $s_i\mathbf{R}s_j$ *for the least* $i \leq k$ *and the least* $j \leq k$ *such that:*

- *for some* $r \in \mathbf{W}(\mathbf{F})$, *both* $s_i\mathbf{R}r$ *and* $r\mathbf{R}s_j$ *are entries on P, and*
- $s_i\mathbf{R}s_j$ *does not yet occur on P.*

Case 3: $n = 4k + 3$, *for* $k \in \mathbb{N}$.

For an **S4F**-*CST, do nothing.*

For an **S4C**-*CST, construct* τ_{n+1} *from* τ_n *by appending, to the end of each non-contradictory path P through* τ_n, *an entry* $\mathbf{F}(s_i)\,\mathbf{R}\,\mathbf{F}(s_j)$ *for the least* $i \leq k$ *and the least* $j \leq k$ *such that:*

- $s_i\mathbf{R}s_j$ *is an entry on P, and*

- $\mathbf{F}(s_i)\,\mathbf{R}\,\mathbf{F}(s_j)$ *does not yet occur on* P.

Case 4: $n = 4k$, *for* $k \in \mathbb{N}$.

Let e be the $<_{LL}$ *least node of* τ_n *such that e is an occurrence of a signed forcing assertion E that is not reduced on some non-contradictory path* P *through* τ_n.

Sub-case 4(i). *E is* not *of the form* $T[\ t \Vdash \Box\psi\ ]$. *Then* τ_{n+1} *is the tableaux obtained from* τ_n *by appending to the end of every non-contradictory path* P *through* τ_n *on which e is not reduced, the atomic tableaux with root entry E, using the appropriate case of the* (**Develop**) *rule. This atomic tableaux is uniquely specified by requiring that in case* $(F\Box)$, *where E is* $F[\ t \Vdash \Box\psi\ ]$, *we use the entries* $t\mathbf{R}\mathbf{w}_i$ *and* $F[\ \mathbf{w}_i \Vdash \psi\ ]$, *where i is the least* $j \in \mathbb{N}$ *such that* $\mathbf{w}_j$ *does not occur on* τ_n.

Sub-case 4(ii). *E is of the form* $T[\ t \Vdash \Box\psi\ ]$, *e is occurrence m of E on* P, *and* $m = p(j, l)$. *Then* τ_{n+1} *is the tableaux obtained from* τ_n *by appending to the end of every non-contradictory path* P *through* τ_n *on which e is not reduced, the atomic tableaux with root entry E, as determined by the* $(T\Box)(s_j)$ *case of the* (**Develop**) *rule.*

For example, the tableaux in Figure 4.3 for the instance $[a]\Box p \to \Box[a]p$ of the continuity axiom depicts (without repetition of developed entries) an initial subtree of the **S4F**-CST for that formula. The sole path P continues with reflexive accessibility assertions $\mathbf{F}^k(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^k(\mathbf{w}_i)$ for $i = 0, 1$, and for all $k \in \mathbb{N}$; the transitivity rule does not add any new entries; and the $T\Box$ entry $T[\ \mathbf{F}(\mathbf{w}_0) \Vdash \Box p\ ]$ would be repeated infinitely often, without any effect since no new entries $\mathbf{F}(\mathbf{w}_0)\,\mathbf{R}\,s_j$ will ever be added.

Lemma 4.4.4 *Let* τ *be a CST and* P *a path through* τ.

(i) *If* τ *is an* **S4F**-*CST and* $\mathbf{F}^k(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^l(\mathbf{w}_j)$ *is an entry on* P, *then* $i < j$ *and* $l = 0$.

(ii) *If* τ *is an* **S4C**-*CST and* $\mathbf{F}^k(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^l(\mathbf{w}_j)$ *is an entry on* P, *then* $i < j$, $l \leq k$ *and for* $k - l \leq r \leq l$, $\mathbf{F}^{k-l+r}(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}^r(\mathbf{w}_j)$ *is an entry on* P.

Proof. Requires an analysis of how accessibility assertions are added to a path using the (**R-Reflex**), (**R-Trans**) or (**F-Cont**) rules, or using the $(F\Box)$ case of the (**Develop**) rule, similar to that in the proofs of Propositions 4.2.7 and 4.2.8. ■

Proposition 4.4.5 *For each formula* φ *of* $\mathcal{L}_{\Box a}$, *the* **S4F**-*CST* (**S4C**-*CST*) τ^φ *is finished.*

Proof. Fix $\varphi \in \mathcal{L}_{\Box a}$, and let $\tau = \tau^{\varphi} = \bigcup_{n \in I} \tau_n$, for some $I \preceq \mathbb{N}$. Suppose there is an occurrence e of a signed forcing assertion E and a non-contradictory path P through τ, such that e is *not* reduced on P. Suppose t is the subject of E, e is occurrence $m = p(j, l)$ of E, and there are q nodes of τ that are $<_{LL} e$. Let n be large enough so that:

(a) the occurrence e of E is on the path $\mathrm{P}_n = \mathrm{P} \lceil \tau_n$; and

(b) the entry $t\mathbf{R}s_j$ is on P_n if it is on P at all.

Then from the definition of the CST, it is clear that we must reduce e on P by the time we form τ_{n+4q+1}, and once e is reduced, it will remain reduced by Lemma 4.4.2. So τ is finished, as required. ■

Corollary 4.4.6 *Let $\tau^{\varphi} = \bigcup_{n \in I} \tau_n$ be the* **S4F**-*CST (***S4C**-*CST) for some $\varphi \in \mathcal{L}_{\Box a}$, let P be a non-contradictory path through τ^{φ}, let $t \in \mathbf{W}(\mathbf{F})$ and let ψ be a subformula of φ.*

If there is an occurrence 0 on P of the signed forcing assertion $T[\, t \Vdash \Box\psi \,]$, then there is an occurrence m on P of $T[\, t \Vdash \Box\psi \,]$, for every $m \in \mathbb{N}$.

Proof. Immediate from Theorem 4.4.5. For each $j, l \in \mathbb{N}$, an "l^{th}" attempt will be made to develop occurrence $m = p(j, l)$ of $T[\, t \Vdash \Box\psi \,]$ on P using the $(T\Box)(s_j)$ case of the (**Develop**) rule, so generating occurrence $m + 1$ on P. ■

Theorem 4.4.7 *If $\tau = \tau^{\varphi} = \bigcup_{n \in I} \tau_n$ is the* **S4F**-*CST (***S4C**-*CST) for $\varphi \in \mathcal{L}_{\Box a}$, P is a non-contradictory path through τ^{φ}, $\mathcal{K}_P = (W_P, R_P, F_P)$ is the term frame of P, and η_P is the path valuation for $\mathcal{K}_P$ (Definition 4.2.4), then*

(i) *for all $t, s \in W_P$,*

$$t\mathbf{R}s \text{ is an entry on P} \quad \Leftrightarrow \quad (t, s) \in R_P$$

(ii) *for all $t \in W_P$ and all $\psi \in \mathcal{L}_{\Box a}$,*

$$\begin{array}{lcl} T[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \Vdash^{\mathcal{K}_P}_{\eta_P} \psi \\ F[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \nVdash^{\mathcal{K}_P}_{\eta_P} \psi \end{array}$$

Hence the identity function on W_P witnesses that $(\mathcal{K}_P, \eta_P)$ agrees with P.

Proof. For (i), recall from Definition 4.2.1 that for **S4F** tableaux

$$R_{\mathrm{P}} = \bigcup_{m\in\mathrm{N}} R_m$$

while from Definition 4.2.2 for **S4C** tableaux

$$R_{\mathrm{P}} = \bigcup_{m\in\mathrm{N}} R_m^+$$

where

$$\begin{array}{lllll} R_0 & = & R_0^+ & = \{(t,t) \mid t \in W_{\mathrm{P}}\} \\ R_1 & = & R_1^+ & = \{(t,s) \in W_{\mathrm{P}} \times W_{\mathrm{P}} \mid t\mathbf{R}s \text{ is an entry on P}\} \end{array}$$

and for **S4F** tableaux,

$$R_{m+1} = R_m \cup \{(t,s) \mid (\exists r \in W_{\mathrm{P}})\ (t,r) \in R_m \text{ and } (r,s) \in R_m\}$$

while for **S4C** tableaux

$$\begin{array}{lll} R_{m+1}^+ & = & R_m^+ \cup \{(t,s) \mid (\exists r \in W_{\mathrm{P}})\ (t,r) \in R_m^+ \text{ and } (r,s) \in R_m^+\} \\ & & \cup\{(\mathbf{F}(t),\mathbf{F}(s)) \mid (t,s) \in R_m^+\} \end{array}$$

Hence in either case, we have trivially,

$$t\mathbf{R}s \text{ is an entry on P} \quad \Rightarrow \quad (t,s) \in R_{\mathrm{P}}$$

Dealing first with the reflexive accessibility assertions, fix $t \in W_{\mathrm{P}}$, say $t = \mathbf{F}^m(\mathbf{w}_l)$. By definition of W_{P}, $\mathbf{w}_l$ is the subject of some signed forcing assertion on t on P; let $S[\ \mathbf{w}_l \Vdash \psi\ ]$ be the first signed forcing assertion on P having $\mathbf{w}_l$ as its subject, for any $\psi \in \mathcal{L}_{\Box a}$. Let $j = p(l,m)$, so $t = s_j$ in the fixed enumeration of world terms. Let k the least number such that $k \geq j$ and $S[\ \mathbf{w}_l \Vdash \psi\ ]$ is an entry on $\mathrm{P}_{4k+1} = \mathrm{P}\lceil\ \tau_{4k+1}$. Then by the definition of **S4F**-CST (**S4C**-CST), the entry $t\mathbf{R}t$ will be added to P by stage $n = 4k+1$ of the construction, if it is not already there.

For the transitive and **F**-functional closure, we prove by induction on $m \geq 1$ that

$$(t,s) \in R_m^+ \quad \Rightarrow \quad t\mathbf{R}s \text{ is an entry on P}$$

for all $t, s \in W_{\mathrm{P}}$. Since $R_m \subseteq R_m^+$, the induction can be trivially modified for the transitive closure alone.

For the base case of $m = 1$, the result is immediate. Assume the result holds for $m \geq 1$, and fix $t, s \in W_{\mathrm{P}}$. For the inductive step, suppose $(t,s) \in R_{m+1}^+ - R_m^+$. There are two cases.

Case (a): there exists $r \in W_{\mathrm{P}}$ such that $(t,r) \in R_m^+$ and $(r,s) \in R_m^+$. Hence by the induction hypothesis, $t\mathbf{R}r$ and $r\mathbf{R}s$ are both entries on P. For some $i,j \in \mathbb{N}$, $t = s_i$ and $s = s_j$ in the fixed enumeration of world terms. Let k be the least number such that $k \geq \max\{i,j\}$, and $s_i\mathbf{R}r$ and $r\mathbf{R}s_j$ are both entries on $\mathrm{P}_{4k+2} = \mathrm{P}\upharpoonright \tau_{4k+2}$. Then by the definition of **S4F**-CST (**S4C**-CST), the entry $t\mathbf{R}s$ (i.e. $s_i\mathbf{R}s_j$) will be added to P by stage $n = 4k+2$ of the construction, if it is not already there.

Case (b): there exists $t', s' \in W_{\mathrm{P}}$ such that $t = \mathbf{F}(t')$, $s = \mathbf{F}(s')$ and $(t',s') \in R_m^+$. Hence by the induction hypothesis, $t'\mathbf{R}s'$ is an entry on P. For some $i,j \in \mathbb{N}$, $t' = s_i$ and $s' = s_j$ in the fixed enumeration of world terms. Let k be the least number such that $k \geq \max\{i,j\}$, and $s_i\mathbf{R}s_j$ is an entry on $\mathrm{P}_{4k+3} = \mathrm{P}\upharpoonright \tau_{4k+3}$. Then by the definition of **S4C**-CST, the entry $t\mathbf{R}s$ (i.e. $\mathbf{F}(s_i)\,\mathbf{R}\,\mathbf{F}(s_j)$) will be added to P by stage $n = 4k+3$ of the construction, if it is not already there.

Hence

$$t\mathbf{R}s \text{ is an entry on P} \quad \Leftrightarrow \quad (t,s) \in R_{\mathrm{P}}$$

as required.

For (ii), we proceed by induction on the complexity of formulas $\psi \in \mathcal{L}_{\Box a}$ to prove that for all $t \in W_{\mathrm{P}}$,

$$\begin{array}{lcl} T[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \Vdash_{\eta_{\mathrm{P}}} \psi \\ F[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \nVdash_{\eta_{\mathrm{P}}} \psi \end{array}$$

For atomic propositions $p \in AP$, fix $t \in W_{\mathrm{P}}$. we have from the definition of η_{P} that

$$T[\, t \Vdash p \,] \text{ is an entry on P} \quad \Leftrightarrow \quad t \Vdash_{\eta_{\mathrm{P}}} p$$

If $F[\, t \Vdash \psi \,]$ is an entry on P, then since P is non-contradictory, $T[\, t \Vdash \psi \,]$ is not an entry on P, hence $t \nVdash_{\eta_{\mathrm{P}}} p$.

For the inductive cases, the key result is that every occurrence of every signed forcing assertion on P is reduced, since by Theorem 4.4.5, the **S4F**-CST (**S4C**-CST) τ is finished.

For the Boolean connectives $\neg$ and $\rightarrow$, the induction steps are completely trivial.

For $\Box$, assume by induction that the result holds for ψ and all world terms in W_{P}. Fix $t \in W_{\mathrm{P}}$ and suppose $T[\, t \Vdash \Box\psi \,]$ is an entry on P. By Corollary 4.4.6, for each pair $j,l \in \mathbb{N}$, there is an occurrence $m = p(j,l)$ on P of $T[\, t \Vdash \Box\psi \,]$, and the reduction of this occurrence will consist of the "l^{th}" attempt to develop the entry using the $(T\Box)(s_j)$ case of the (**Develop**) rule. Hence if $t\mathbf{R}s_j$ ever appears on P, then $T[\, s_j \Vdash \psi \,]$ will be an entry on P. Hence by the induction hypothesis applied to s_j, we have $s_j \Vdash_{\eta_{\mathrm{P}}} \psi$. Since by (i), $t\mathbf{R}s_j$ is an entry on P iff $(t,s_j) \in R_{\mathrm{P}}$, we have by the definition of forcing that $t \Vdash_{\eta_{\mathrm{P}}} \Box\psi$.

On the other hand, suppose $F[\ t \Vdash \Box\psi\ ]$ is an entry on P. Then since (every occurrence of) this entry is reduced, we will have both $t\mathbf{R}\mathbf{w}_i$ and $F[\ \mathbf{w}_i \Vdash \psi\ ]$ added as entries on P at some stage n in the construction of τ, where the world symbol $\mathbf{w}_i$ does not previously occur in any entry in τ_n. By (i), $t\mathbf{R}\mathbf{w}_i$ is an entry on P iff $(t, \mathbf{w}_i) \in R_{\mathrm{P}}$, so by the definition of forcing we have that $t \nVdash_{\eta_{\mathrm{P}}} \Box\psi$.

For $[a]$, recall that in the term frame $\mathcal{K}_{\mathrm{P}} = (W_{\mathrm{P}}, R_{\mathrm{P}}, F_{\mathrm{P}})$, the function F_{P} is just the term constructor $t \mapsto \mathbf{F}(t)$. Assume by induction that the result holds for ψ and all world terms in W_{P}. Fix $t \in W_{\mathrm{P}}$ and suppose $T[\ t \Vdash [a]\psi\ ]$ is an entry on P. Then since (every occurrence of) this entry is reduced, $T[\ \mathbf{F}(t) \Vdash \psi\ ]$ is an entry on P. Hence by the induction hypothesis applied to $\mathbf{F}(t)$, we have $F_{\mathrm{P}}(t) \Vdash_{\eta_{\mathrm{P}}} \psi$. Then by the definition of forcing, $t \Vdash_{\eta_{\mathrm{P}}} [a]\psi$.

On the other hand, suppose $F[\ t \Vdash [a]\psi\ ]$ is an entry on P. Then since (every occurrence of) this entry is reduced, $F[\ \mathbf{F}(t) \Vdash \psi\ ]$ is an entry on P. Hence by the induction hypothesis applied to $\mathbf{F}(t)$, we have $F_{\mathrm{P}}(t) \nVdash_{\eta_{\mathrm{P}}} \psi$. Then by the definition of forcing, $t \nVdash_{\eta_{\mathrm{P}}} [a]\psi$. ■

Theorem 4.4.8 Kripke completeness of **S4F** and **S4C** tableaux

For each formula φ of $\mathcal{L}_{\Box a}$, if **S4F** $\nvdash_T \varphi$ *(***S4C** $\nvdash_T \varphi$ *), then there is a (continuous) countable partially-ordered Kripke frame $\mathcal{K}$ and a valuation η for $\mathcal{K}$ such that $(\mathcal{K}, \eta) \nVdash \varphi$.*

Proof. Consider the **S4F**-CST (**S4C**-CST) $\tau = \tau^{\varphi}$ for φ with root entry $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$. If **S4F** $\nvdash_T \varphi$ (**S4C** $\nvdash_T \varphi$), then there is a non-contradictory path P through τ. Let $\mathcal{K}_{\mathrm{P}} = (W_{\mathrm{P}}, R_{\mathrm{P}}, F_{\mathrm{P}})$ be the term frame for P and let η_{P} be the canonical valuation for $\mathcal{K}_{\mathrm{P}}$. By Proposition 4.2.7 (Proposition 4.2.8), W_{P} is countable, R_{P} is a partial order, and F_{P} is injective. By Theorem 4.4.7, $(\mathcal{K}_{\mathrm{P}}, \eta_{\mathrm{P}})$ agrees with P, witnessed by the identity map. Hence $\mathbf{w}_0 \nVdash^{\mathcal{K}_{\mathrm{P}}}_{\eta_{\mathrm{P}}} \varphi$, and so $(\mathcal{K}_{\mathrm{P}}, \eta_{\mathrm{P}}) \nVdash \varphi$. ■

4.5 Finite Quotients and Decidability

For each $\varphi \in \mathcal{L}_{\Box a}$, the construction of the complete systematic **S4F** (or **S4C**) tableaux with root entry $F[\ \mathbf{w}_0 \Vdash \varphi\ ]$ is a deterministic procedure. If φ is **S4F** (**S4C**) tableaux provable, then the CST construction will terminate with a finite proof, but if φ is not **S4F** (**S4C**) tableaux provable, then the result will, in general, be an infinite tableaux. We prove the finite model property, and thus decidability, by defining a quotient $\mathcal{K}^{\#}_{\mathrm{P}}$ of the term frame $\mathcal{K}_{\mathrm{P}}$ such that $\left|W^{\#}_{\mathrm{P}}\right| \leq 3^n$, where n is the number of subformulas of φ.

Definition 4.5.1 *For each $\varphi \in \mathcal{L}_{\Box a}$, let $SF(\varphi)$ denote the set of all subformulas of φ. Define the set of* signed subformulas *of φ, $SSF(\varphi)$, by:*

$$SSF(\varphi) \doteq \{ T[\psi] \mid \psi \in SF(\varphi)\} \cup \{ F[\psi] \mid \psi \in SF(\varphi)\}$$

Let τ be the **S4F**-*CST or the* **S4C**-*CST for $\varphi \in \mathcal{L}_{\Box a}$.*
For each world term $t \in \mathbf{W}(\mathbf{F})$ and path P through τ, define

$$\begin{array}{rcl} S_{\mathrm{P}}(t) & \doteq & \{ T[\psi] \mid T[\, t \Vdash \psi \,] \text{ is an entry on } \mathrm{P} \} \\ & \cup & \{ F[\psi] \mid F[\, t \Vdash \psi \,] \text{ is an entry on } \mathrm{P} \} \end{array}$$

A subset $S \subseteq SSF(\varphi)$ of signed subformulas of φ is called inconsistent *if there is a $\psi \in SF(\varphi)$ such that both $T[\psi] \in S$ and $F[\psi] \in S$; and* consistent *otherwise.*

Note that the empty set is consistent. A path P through a CST is non-contradictory iff for each $t \in W_{\mathrm{P}}$, the set $S_{\mathrm{P}}(t)$ is consistent. Note also that terms $t = \mathbf{F}^k(\mathbf{w}_i)$ where $k > [a]\text{-rank}(\varphi)$ will always have $S_{\mathrm{P}}(t) = \emptyset$.

If the cardinality $|SF(\varphi)| = n$, then $|SSF(\varphi)| = 2n$. For each $t \in \mathbf{W}(\mathbf{F})$ and path P through the CST τ^{φ}, we clearly have:

$$S_{\mathrm{P}}(t) \subseteq SSF(\varphi)$$

hence there is at most $2^{2n} = 4^n$ possibilities for $S_{\mathrm{P}}(t)$. Moreover, for subsets $S \subseteq SSF(\varphi)$, if $n + 1 \leq |S| \leq 2n$, then S is inconsistent. By simple combinatorics, the number of consistent subsets of cardinality k is $2^k \binom{n}{k}$. Hence the total number of consistent subsets $S \subseteq SSF(\varphi)$ is:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Lemma 4.5.2 *Let τ be the* **S4F**-*CST or the* **S4C**-*CST for $\varphi \in \mathcal{L}_{\Box a}$, suppose P is a non-contradictory path through τ, and let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P. Then for all $\psi, \chi \in \mathcal{L}_{\Box a}$, $p \in AP$ and $t \in W_P$:*

- *if* $T[p] \in S_{\mathrm{P}}(t)$ *then* $F[p] \notin S_{\mathrm{P}}(t)$;
- *if* $F[p] \in S_{\mathrm{P}}(t)$ *then* $T[p] \notin S_{\mathrm{P}}(t)$;
- *if* $T[\neg\psi] \in S_{\mathrm{P}}(t)$ *then* $F[\psi] \in S_{\mathrm{P}}(t)$;
- *if* $F[\neg\psi] \in S_{\mathrm{P}}(t)$ *then* $T[\psi] \in S_{\mathrm{P}}(t)$;

- *if* $T[\psi \to \chi] \in S_{\mathrm{P}}(t)$ *then* $F[\psi] \in S_{\mathrm{P}}(t)$ *or* $T[\chi] \in S_{\mathrm{P}}(t)$*;*
- *if* $F[\psi \to \chi] \in S_{\mathrm{P}}(t)$ *then* $T[\psi] \in S_{\mathrm{P}}(t)$ *and* $F[\chi] \in S_{\mathrm{P}}(t)$*;*
- *if* $T[\Box\psi] \in S_{\mathrm{P}}(t)$ *then for all* $s \in W_P$*, if* $(t,s) \in R_P$ *then* $T[\psi] \in S_{\mathrm{P}}(s)$*;*
- *if* $F[\Box\psi] \in S_{\mathrm{P}}(t)$ *then for some* $\mathbf{w}_j \in W_P$*,* $(t, \mathbf{w}_j) \in R_P$ *and* $F[\psi] \in S_{\mathrm{P}}(\mathbf{w}_j)$*;*
- *if* $T[[a]\psi] \in S_{\mathrm{P}}(t)$ *then* $T[\psi] \in S_{\mathrm{P}}(\mathbf{F}(t))$*;*
- *if* $F[[a]\psi] \in S_{\mathrm{P}}(t)$ *then* $F[\psi] \in S_{\mathrm{P}}(\mathbf{F}(t))$.

Proof. The clauses for atomic propositions follow from the fact that $S_{\mathrm{P}}(t)$ is consistent since the path P is non-contradictory. The other clauses are essentially a translation into the notation $S_{\mathrm{P}}(t)$ of Theorem 4.4.5 that every **S4F**-CST or **S4C**-CST is *finished*, so every occurrence on P of a signed forcing assertion is reduced on P, together with:

$$(t,s) \in R_{\mathrm{P}} \;\Leftrightarrow\; t\mathbf{R}s \text{ is an entry on P}$$

from Theorem 4.4.7,(i). ■

The content of the lemma is that, in the language of [Fi83], the family of finite sets of signed formulas $\{S_{\mathrm{P}}(t) \mid t \in W_{\mathrm{P}}\}$ is a *consistency property*; the term *Hintikka structure* is also commonly used (although usually for *unsigned* formulas).

Definition 4.5.3 *Let* τ *be the* **S4F**-*CST or the* **S4C**-*CST for* $\varphi \in \mathcal{L}_{\Box a}$*, let* P *be a path through* τ*, and let* $\mathcal{K}_P = (W_P, R_P, F_P)$ *be the term frame for* P*, with* η_P *its path valuation.*

Define an equivalence relation $\equiv_{\mathrm{P}}$ *on* W_P *by:*

$$t \equiv_{\mathrm{P}} s \quad \textit{iff} \quad S_{\mathrm{P}}(t) = S_{\mathrm{P}}(s)$$

Let $\bar{t} = \{s \in W_P \mid t \equiv_{\mathrm{P}} s\}$ *and let* $W_P^{\#}$ *denote the set of all* $\equiv_{\mathrm{P}}$*-equivalence classes* $\bar{t}$*. Define* $\mathcal{K}_P^{\#} = (W_P^{\#}, R_P^{\#}, F_P^{\#})$ *to be the* minimal quotient *(Lemma 3.3.2) under the surjective map* $h : W_P \to W_P^{\#}$ *given by* $h(t) = \bar{t}$*; i.e. for all* $t, s \in W_P$*,*

$$h(t) = h(s) \quad \Leftrightarrow \quad S_{\mathrm{P}}(t) = S_{\mathrm{P}}(s)$$

and

$$(h(t), h(s)) \in R_P^{\#} \quad \Leftrightarrow \quad (t,s) \in R_P$$

and

$$F_P^{\#}(h(t)) = h(F_P(t)) = h(\mathbf{F}(t))$$

Theorem 4.5.4 Finite model property for **S4F** and **S4C**.

Let τ be the **S4F**-*CST or the* **S4C**-*CST for $\varphi \in \mathcal{L}_{\Box a}$, with root entry $F[\, \mathbf{w}_0 \Vdash \varphi \,]$, and suppose P is a non-contradictory path through τ. Let $\mathcal{K}_P = (W_P, R_P, F_P)$ be the term frame for P, with η_P its canonical valuation, and let $\mathcal{K}_P^\# = (W_P^\#, R_P^\#, F_P^\#)$ be the minimal quotient of $\mathcal{K}_P$ under $\equiv_P$. Then:*

(a) *The equivalence relation $\equiv_P$ is of finite index: if $n = |SF(\varphi)|$ then $\left|W_P^\#\right| \leq 3^n$.*

(b) *The induced valuation $\eta_P^\# : W_P^\# \to \mathcal{P}(AP)$ given by:*

$$\eta_P^\#(h(t)) = \eta_P(t) = \{p \in AP \mid T[\, t \Vdash p \,] \text{ is an entry on P}\}$$

is well-defined.

(c) *For all $\psi \in \mathcal{L}_{\Box a}$ and $t \in W_P$,*

$$h(t) \Vdash_\# \psi \quad \Leftrightarrow \quad t \Vdash \psi$$

where $\Vdash$ abbreviates $\Vdash_{\eta_P}^{\mathcal{K}_P}$ and $\Vdash_\#$ abbreviates $\Vdash_{\eta_P^\#}^{\mathcal{K}_P^\#}$.

Hence $(\mathcal{K}_P^\#, \eta_P^\#)$ agrees with P, and in particular, $h(\mathbf{w}_0) \nVdash_\# \varphi$, so

$$(\mathcal{K}_P^\#, \eta_P^\#) \nVdash \varphi$$

Proof. For (a), observe that each equivalence class $h(t) \in W_P^\#$ is associated with the set $S_P(t)$ of signed subformulas of φ, where $S_P(t) \neq S_P(s)$ iff $h(t) \neq h(s)$, and since P is non-contradictory, each $S_P(t)$ is consistent. Since there are 3^n consistent subsets of $SSF(\varphi)$, we have $\left|W_P^\#\right| \leq 3^n$.

For (b), note that the canonical valuation η_P satisfies

$$\eta_P(t) = \{p \in AP \mid T[p] \in S_P(t)\}$$

Hence $h(t) = h(s)$ implies $\eta_P(t) = \eta_P(s)$ and so $\eta_P^\#(h(t)) = \eta_P^\#(h(s))$.

For (c), we proceed by induction on formulas $\psi \in \mathcal{L}_{\Box a}$ to show that for all $t \in W_P$,

$$h(t) \Vdash_\# \psi \quad \Leftrightarrow \quad t \Vdash \psi$$

For atomic propositions $p \in AP$, we have

$$h(t) \Vdash_\# p \quad \Leftrightarrow \quad p \in \eta_P^\#(h(t)) = \eta_P(t) \quad \Leftrightarrow \quad t \Vdash p$$

and P is non-contradictory, so the result is immediate.

The inductive clauses involve only mechanical appeals to the definition of forcing together with the definition of the quotient $\mathcal{K}_{\mathrm{P}}^{\#}$.

For example, for the $\Box$ case, assume by induction the result holds for ψ and all $s \in W_{\mathrm{P}}$. Then

$$\begin{array}{lll} & h(t) \Vdash_{\#} \Box\psi & \\ \Leftrightarrow & \text{for all } s \in W_{\mathrm{P}}, \text{ if } \ ((h(t), h(s)) \in R_{\mathrm{P}}^{\#} \text{ then } \ h(s) \Vdash_{\#} \psi & (1) \\ \Leftrightarrow & \text{for all } s \in W_{\mathrm{P}}, \text{ if } \ (t, s) \in R_{\mathrm{P}} \text{ then } \ s \Vdash \psi & (2) \\ \Leftrightarrow & t \Vdash \Box\psi & (3) \end{array}$$

where the equivalence (1) is from the definition of $W_{\mathrm{P}}^{\#} = \{h(s) \mid s \in W_{\mathrm{P}}\}$ and forcing in $\mathcal{K}_{\mathrm{P}}^{\#}$ for $\Box$ formulas; (2) is from the definition of $R_{\mathrm{P}}^{\#}$ and the induction hypothesis; and (3) is just forcing in $\mathcal{K}_{\mathrm{P}}$ for $\Box$ formulas.

Similarly, for the $[a]$ case, assume by induction the result holds for ψ and all $s \in W_{\mathrm{P}}$. Then

$$\begin{array}{lll} & h(t) \Vdash_{\#} [a]\psi & \\ \Leftrightarrow & F_{\mathrm{P}}^{\#}(h(t)) \Vdash_{\#} \psi & (1) \\ \Leftrightarrow & h(F_{\mathrm{P}}(t)) \Vdash_{\#} \psi & (2) \\ \Leftrightarrow & F_{\mathrm{P}}(t) \Vdash \psi & (3) \\ \Leftrightarrow & t \Vdash [a]\psi & (4) \end{array}$$

where (1) is forcing in $\mathcal{K}_{\mathrm{P}}^{\#}$ for $[a]$ formulas; (2) is from the equation of $F_{\mathrm{P}}^{\#} \circ h = h \circ F_{\mathrm{P}}$; (3) is the induction hypothesis applied to $s = F_{\mathrm{P}}(t)$; and (4) is just forcing in $\mathcal{K}_{\mathrm{P}}$ for $[a]$ formulas.

From Theorem 4.4.7, (ii), we have

$$\begin{array}{lll} T[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \Vdash \psi \\ F[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \nVdash \psi \end{array}$$

Hence

$$\begin{array}{lll} T[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & h(t) \Vdash_{\#} \psi \\ F[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & h(t) \nVdash_{\#} \psi \end{array}$$

and so the quotient map h witnesses that $(\mathcal{K}_{\mathrm{P}}^{\#}, \eta_{\mathrm{P}}^{\#})$ agrees with P. ■

Corollary 4.5.5 *The logics* **S4F** *and* **S4C** *are decidable.*

$(\mathcal{K}_{\mathrm{P}}^{\#}, \eta_{\mathrm{P}}^{\#})$ is most certainly a near relative of the finite model one would get if one took a filtration through set of subformulas $SF(\varphi)$ of the canonical maximal-consistent sets model $(\mathcal{K}_0, \eta_0)$ of Propositions 2.3.7 and 3.2.3.

Corollary 4.5.6 *For each formula φ of $\mathcal{L}_{\Box a}$, the following are equivalent:*

(1.) $\mathbf{S4F(S4C)} \vdash_T \varphi$

(2.) $\mathbf{S4F(S4C)} \vdash_H \varphi$

(3.) $\mathfrak{T} \models \varphi$ *for all (continuous) topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$,

(4.) $\mathfrak{T} \models \varphi$ *for all (continuous) D-topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$,

(5.) $\mathcal{K} \Vdash \varphi$ *for all (continuous) Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$,

(6.) $\mathfrak{T} \models \varphi$ *for all (continuous) countable* T_0 *D-topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$,

(7.) $\mathcal{K} \Vdash \varphi$ *for all (continuous) countable partially-ordered Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$;

(8.) $\mathcal{K} \Vdash \varphi$ *for all (continuous) finite Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$;

Proof. (8.) $\Rightarrow$ (1.) is Theorem 4.5.4, and (5.) $\Rightarrow$ (8.) is trivial. (7.) $\Rightarrow$ (1.) is the completeness theorems for **S4F** (**S4C**) tableaux, in Theorem 4.4.8. (1.) $\Rightarrow$ (5.) is the Kripke soundness of **S4F** (**S4C**) tableaux, in Theorem 4.3.3. (3.) $\Rightarrow$ (4.) and (5.) $\Rightarrow$ (7.) are trivial. (4.) $\Leftrightarrow$ (5.) and (6.) $\Leftrightarrow$ (7.) are Corollaries 2.4.13 (3.1.6) and 2.4.14 (3.1.7). (2.) $\Rightarrow$ (3.) is the topological soundness of the Hilbert-style proof system, in Proposition 2.2.2 (3.2.2). And (5.) $\Rightarrow$ (2.) is the Kripke completeness results for the Hilbert-style proof system, in Proposition 2.3.7 (3.2.3). In summary,

$$\begin{array}{l}
(5.) \Rightarrow (8.) \Rightarrow (1.) \Rightarrow (5.) \\
(5.) \Rightarrow (2.) \Rightarrow (3.) \Rightarrow (4.) \Leftrightarrow (5.) \\
(5.) \Rightarrow (7.) \Rightarrow (1.) \Rightarrow (5.) \\
(6.) \Leftrightarrow (7.)
\end{array}$$

■

Chapter 5

Topological Propositional Dynamic Logic TPDL

5.1 Syntax and Topological Semantics

S4C is the logic of *one* continuous action, and although not without interest in its own right, its purpose is primarily to provide a solid foundation. We need to be able to talk about more actions, and we need to be able to combine them in interesting ways. To this effect, we create a *modal*, **S4**-based, dynamic logic by overlaying the apparatus of propositional dynamic logic ([FL79], [Pra79], [Par81], [Seg82]).

In this setting, atomic actions $a \in \Sigma$ will be interpreted by continuous total functions, and compound actions $\alpha \in Act(\Sigma)$ are generated using the Kleene operations of composition, sum (non-deterministic choice) and iteration (star). The "test" operation is omitted at this stage, pending a further clarification of an appropriate semantics. So what is overlaid on **S4C** is actually the test-free fragment of *deterministic* propositional dynamic logic **DPDL**, further restricted to atomic actions whose interpretations are both *functional* (deterministic) and *total*. **DPDL** is studied in [BHP82]. A precursor can be found in [Con77], where atomic commands are interpreted by partial functions. Within the "algorithmic logic" school of Salwicki and Mirkowska, the logic of deterministic total actions is briefly studied in [MS87], Chp.V, §8.

Very recent work of Kremer, Mints and Rybakov (see the abstracts [KrMi97], [Kre97], and [KrMiR97]) examines a family of logics **DTL** (Dynamic Topological Logics) extending **S4** by the addition of a "next" operator $\bigcirc$ corresponding to our $[a]$ modality, for a single atomic action a, and a "star" operator $\star$ corresponding to $[a^*]$. The abstract announces axiomatizations of various fragments; for example, the star-free fragment of the logic $\mathbf{DTL}_{\mathcal{H}}$ of *homeomorphic* functions.

Definition 5.1.1 *Let $\Phi = AP$ be a countable set of* atomic propositions, *and let Σ be a countable set of* atomic actions. *The set of* formulas *$Form(\Phi, \Sigma)$ and the set of* action expressions *(or more simply,* actions*) $Act(\Sigma)$ of the language $\mathcal{L}_{\Box}(\Phi, \Sigma) = Form(\Phi, \Sigma) \cup Act(\Sigma)$ are defined inductively as follows:*

- *if $a \in \Sigma$ then $a \in Act(\Sigma)$;*
- *if $\alpha, \beta \in Act(\Sigma)$ then $(\alpha\beta) \in Act(\Sigma)$, $(\alpha + \beta) \in Act(\Sigma)$ and $(\alpha^*) \in Act(\Sigma)$;*
- *if $p \in \Phi$ then $p \in Form(\Phi, \Sigma)$;*
- *if $\varphi, \psi \in Form(\Phi, \Sigma)$ then $\neg\varphi \in Form(\Phi, \Sigma)$, $\varphi \rightarrow \psi \in Form(\Phi, \Sigma)$ and $\Box\varphi \in Form(\Phi, \Sigma)$;*
- *if $\varphi \in Form(\Phi, \Sigma)$ and $\alpha \in Act(\Sigma)$ then $\langle\alpha\rangle\varphi \in Form(\Phi, \Sigma)$.*

We omit parentheses when no confusion results.

For each $\alpha \in Act(\Sigma)$, define:

$$[\alpha]\varphi \doteq \neg\langle\alpha\rangle\neg\varphi$$

and define the other Boolean connectives and constants, as well as the **S4** $\Diamond$ operator, as in Section 2.1 above.

The intended intuitive reading of formulas is:

$$\begin{array}{lcl} [\alpha]\varphi & \sim & \text{“action } \alpha \text{ \textit{always} makes it the case that } \varphi\text{”} \\ \langle\alpha\rangle\varphi & \sim & \text{“action } \alpha \text{ \textit{sometimes} makes it the case that } \varphi\text{”} \end{array}$$

with “*sometimes* = *always*” for atomic actions $a \in \Sigma$, and more generally, for compositions of atomic actions. The intended reading of action expressions is:

$$\begin{array}{rcl} \alpha\beta & \sim & \text{“perform action } \alpha\text{, then action } \beta\text{”} \\ \alpha + \beta & \sim & \text{“perform (non-deterministically) either action } \alpha \text{ or action } \beta\text{”} \\ \alpha^* & \sim & \text{“perform action } \alpha \text{ repeatedly, some finite number of times”} \end{array}$$

A finite sequence (under composition) of atomic actions is word

$$u = (a_1 \cdots a_n) \in \Sigma^*$$

which can be thought of as a *basic control script*. We say more about what can be expressed in the language and logic in Section 5.3 below.

Definition 5.1.2 *Given a topological space $(X, \mathcal{T})$, let $C_{\mathcal{T}}(X)$ denote the set of all total functions $f : X \to X$ that are continuous w.r.t. the topology $\mathcal{T}$.*

Definition 5.1.3 *A (continuous)* topological structure *for the language $\mathcal{L}_{\Box}(\Phi, \Sigma)$ is a triple $\mathfrak{T} = (X, \mathcal{T}, \nu)$ where*

- *$X \neq \emptyset$ is the state space;*
- *$\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on X; and*
- *$\nu : \Sigma \to C_{\mathcal{T}}(X)$ is a map assigning a continuous total function $\nu(a) : X \to X$ to each atomic action $a \in \Sigma$.*

The map ν uniquely extends to a map $\sigma = \sigma_\nu : Act(\Sigma) \to (\mathcal{P}(X) \to \mathcal{P}(X))$, assigning a unary operator $\sigma(\alpha) : \mathcal{P}(X) \to \mathcal{P}(X)$ to each action $\alpha \in Act(\Sigma)$, according to the following inductive clauses, where $a \in \Sigma$, $\alpha, \beta \in Act(\Sigma)$, and $A \in \mathcal{P}(X)$:

$$
\begin{aligned}
(\sigma(a))\,(A) &\doteq \nu(a)^{-1}(A) \\
(\sigma(\alpha\beta))\,(A) &\doteq (\sigma(\alpha) \circ \sigma(\beta))\,(A) \\
(\sigma(\alpha + \beta))\,(A) &\doteq (\sigma(\alpha))\,(A) \cup (\sigma(\beta))\,(A) \\
(\sigma(\alpha^*))\,(A) &\doteq \textstyle\bigcup_{k \in \mathbb{N}} \left(\sigma(\alpha)^k\right)(A)
\end{aligned}
$$

where $\sigma(\alpha)^0 \doteq \mathbf{1}_{\mathcal{P}(X)}$ is the identity operator on $\mathcal{P}(X)$ and $\sigma(\alpha)^{k+1} \doteq \sigma(\alpha) \circ \sigma(\alpha)^k$.

A valuation *for a topological structure $\mathfrak{T} = (X, \mathcal{T}, \nu)$ is any map $\xi : \Phi \to \mathcal{P}(X)$ assigning a subset $\xi(p) \subseteq X$ to each $p \in \Phi$. Each such valuation uniquely extends to a valuation map $\|\cdot\|_\xi : Form(\Phi, \Sigma) \to \mathcal{P}(X)$, assigning a subset $\|\varphi\|_\xi \in \mathcal{P}(X)$ to each formula $\varphi \in Form(\Phi, \Sigma)$, according to the following inductive clauses, where $p \in \Phi$, $\varphi, \psi \in Form(\Phi, \Sigma)$, and $\alpha \in Act(\Sigma)$:*

$$
\begin{aligned}
\|p\|_\xi &\doteq \xi(p) \\
\|\neg\varphi\|_\xi &\doteq -\|\varphi\|_\xi \\
\|\varphi \to \psi\|_\xi &\doteq -\|\varphi\|_\xi \cup \|\psi\|_\xi \\
\|\Box\varphi\|_\xi &\doteq int_{\mathcal{T}}\left(\|\varphi\|_\xi\right) \\
\|\langle\alpha\rangle\varphi\|_\xi &\doteq \sigma(\alpha)\left(\|\varphi\|_\xi\right)
\end{aligned}
$$

A topological model *for $\mathcal{L}_{\Box}(\Phi, \Sigma)$ is a pair $(\mathfrak{T}, \xi)$, where ξ is a valuation for $\mathfrak{T}$.*

For formulas $\varphi \in Form(\Phi, \Sigma)$, the notions of satisfiability *and* truth *in a model $(\mathfrak{T}, \xi)$,* validity *in a structure $\mathfrak{T}$, and* topological validity, *are as in Definition 2.1.5.*

The topological semantics for the defined action modalities $[\alpha]$ are given by:

$$\|[\alpha]\varphi\|_\xi \quad = \quad (-\sigma(\alpha)-)\left(\|\varphi\|_\xi\right)$$

By Boolean duality, we can define a family of unary operators $\pi(\alpha)$ interpreting the $[\alpha]$ modalities which agree with $\sigma(\alpha)$ at the atomic level; i.e. for atomic actions $a \in \Sigma$,

$$\pi(a) = (-\sigma(a)-) = \sigma(a) = \nu(a)^{-1}$$

Proposition 5.1.4 *Let $\mathfrak{T} = (X, \mathcal{T}, \nu)$ be a topological structure for $\mathcal{L}_\Box(\Phi, \Sigma)$, and let $\sigma : Act(\Sigma) \to (\mathcal{P}(X) \to \mathcal{P}(X))$ be the operator map induced by ν. For each action $\alpha \in Act(\Sigma)$, define its Boolean dual operator $\pi(\alpha) : \mathcal{P}(X) \to \mathcal{P}(X)$ by:*

$$(\pi(\alpha))(A) \doteq (-\sigma(\alpha)-)(A)$$

Then for $a \in \Sigma$, $\alpha, \beta \in Act(\Sigma)$, $A \in \mathcal{P}(X)$ and $\varphi \in Form(\Phi, \Sigma)$:

$$\begin{aligned}
(\pi(a))(A) &\doteq \nu(a)^{-1}(A) = \sigma(a)(A) \\
(\pi(\alpha\beta))(A) &\doteq (\pi(\alpha) \circ \pi(\beta))(A) \\
(\pi(\alpha + \beta))(A) &\doteq (\pi(\alpha))(A) \cap (\pi(\beta))(A) \\
(\pi(\alpha^*))(A) &\doteq \textstyle\bigcap_{k \in \mathrm{N}} \left(\pi(\alpha)^k\right)(A)
\end{aligned}$$

where $\pi(\alpha)^0 \doteq \mathbf{1}_{\mathcal{P}(X)}$ is the identity operator on $\mathcal{P}(X)$ and $\pi(\alpha)^{k+1} \doteq \pi(\alpha) \circ \pi(\alpha)^k$. Hence for any valuation ξ for $\mathfrak{T}$, we have:

$$\|[\alpha]\varphi\|_\xi \quad = \quad \pi(\alpha)\left(\|\varphi\|_\xi\right)$$

Proof. Straightforward induction on actions α. ■

The remainder of this section is devoted to studying the behavior of $\sigma(\alpha)$ and $\pi(\alpha)$ as operators on the topological Boolean algebra

$$\mathfrak{B}_\mathcal{T}(X) = (\mathcal{P}(X), \cup, \cap, -, X, \emptyset, int_\mathcal{T})$$

of a topological space $(X, \mathcal{T})$, and identifying the sense in which these operators are *continuous* relative to the topology $\mathcal{T}$. The behavior of the operators can be separated into two sorts: how they behave with respect to the Boolean algebra operations, and how they behave with respect to the topological interior operator. The pioneering work on Boolean algebras with operators is Jónsson and Tarski's [JT51].

Definition 5.1.5 [JT51].

Let $(X, \mathcal{T})$ be a topological space, and let $\mathfrak{B}_{\mathcal{T}}(X) = (\mathcal{P}(X), \cup, \cap, -, X, \emptyset, int_{\mathcal{T}})$ its topological Boolean algebra..

A unary operator $H : \mathcal{P}(X) \to \mathcal{P}(X)$ is called normal and completely additive *on $\mathfrak{B}_{\mathcal{T}}(X)$, abbreviated* "nca" *iff for all $A, B, A_i \in \mathcal{P}(X)$,*

(a) $H(\emptyset) = \emptyset$;

(b) $H\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} H(A_i)$;

(c) *if* $A \subseteq B$ *then* $H(A) \subseteq H(B)$;

(d) $H(A \cap B) \subseteq H(A) \cap H(B)$.

Dually, a unary operator $H : \mathcal{P}(X) \to \mathcal{P}(X)$ is called normal and completely multiplicative *on $\mathfrak{B}_{\mathcal{T}}(X)$, abbreviated* "ncm" *iff for all $A, B, A_i \in \mathcal{P}(X)$,*

(a′) $H(X) = X$;

(b′) $H\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} H(A_i)$;

(c′) *if* $A \subseteq B$ *then* $H(A) \subseteq H(B)$;

(d′) $H(A) \cup H(B) \subseteq H(A \cup B)$.

We call a unary operator $H : \mathcal{P}(X) \to \mathcal{P}(X)$ continuous on $\mathfrak{B}_{\mathcal{T}}(X)$ *iff for all $A \in \mathcal{P}(X)$,*

$$H(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(H(A))$$

The terms "normal" and "completely additive" are from [JT51], and refer to properties (a) and (b) respectively. Properties (c) and (d) are trivial consequences of (b), but are included here for useful reference. Observe that if a unary operator $H : \mathcal{P}(X) \to \mathcal{P}(X)$ is nca, then considering $\mathcal{P}(X)$ as a *dcpo* (pointed, directed-complete partial order) under inclusion, then H is trivially *Scott-continuous*, since Scott-continuity only requires preservation of directed unions.

The notion of continuity in $\mathfrak{B}_{\mathcal{T}}(X)$ is an abstraction of the behavior of the inverse-image f^{-1} with respect to the $int_{\mathcal{T}}$ operator when f is continuous with respect to $\mathcal{T}$.

Lemma 5.1.6 *Let $\mathfrak{B}_{\mathcal{T}}(X)$ be the topological Boolean algebra of a topological space $(X, \mathcal{T})$, and let $H : \mathcal{P}(X) \to \mathcal{P}(X)$ be a unary operator which is $\subseteq$-monotone (properties (c) and (c′) in Definition 5.1.5). Then the following are equivalent:*

(i) *H is continuous on $\mathfrak{B}_{\mathcal{T}}(X)$;*

(ii) *for all $A \in \mathcal{P}(X)$, $H\,(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}\,(H\,(int_{\mathcal{T}}(A)))$;*

(iii) *for all open $U \in \mathcal{T}$, $H(U) \in \mathcal{T}$.*

Proof. Going back to the proof of Proposition 3.1.1, the only property of the inverse-image needed to establish these equivalences in the case $H = f^{-1}$ (for total $f : X \to X$) is $\subseteq$-monotonicity. ■

Lemma 5.1.7 *Let $\mathfrak{T} = (X, \mathcal{T}, \nu)$ be a topological structure for $\mathcal{L}_{\Box}(\Phi, \Sigma)$, let $\sigma : Act(\Sigma) \to (\mathcal{P}(X) \to \mathcal{P}(X))$ be the operator map induced by ν, let $\mathfrak{B}_{\mathcal{T}}(X)$ be the topological Boolean algebra of $(X, \mathcal{T})$, and let $\alpha \in Act(\Sigma)$.*

(i) *The operator $\sigma(\alpha)$ is nca and continuous on $\mathfrak{B}_{\mathcal{T}}(X)$.*

(ii) *The operator $\pi(\alpha)$ is ncm, and if $\mathcal{T}$ is a D-topology then $\pi(\alpha)$ is continuous on $\mathfrak{B}_{\mathcal{T}}(X)$.*

(iii) *For all $A \in \mathcal{P}(X)$, $\pi(\alpha)(A) \subseteq \sigma(\alpha)(A)$.*

Proof. The base case of the induction on actions $\alpha \in Act(\Sigma)$ for each of (i), (ii) and (iii) is immediate: for atomic actions $a \in \Sigma$, $\sigma(a) = \pi(a) = f^{-1}$, where $f = \nu(a) : X \to X$ is a total function continuous w.r.t. $\mathcal{T}$. For (i), establishing the nca properties of $\sigma(\alpha)$ is easy. We write out the details of the induction on actions $\alpha \in Act(\Sigma)$ for the continuity of $\sigma(\alpha)$ on $\mathfrak{B}_{\mathcal{T}}(X)$:

$$\sigma(\alpha)\,(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}\,(\sigma(\alpha)(A))$$

The base case for atomic actions is done. For composition,

$$\begin{array}{lll} & \sigma(\alpha\beta)\,(int_{\mathcal{T}}(A)) & \\ = & \sigma(\alpha)\,(\sigma(\beta)\,(int_{\mathcal{T}}(A))) & \\ \subseteq & \sigma(\alpha)\,(int_{\mathcal{T}}\,(\sigma(\beta)(A))) & \text{Ind. Hyp. for } \beta \text{ and } A \\ \subseteq & int_{\mathcal{T}}\,(\sigma(\alpha)\,(\sigma(\beta)(A))) & \text{Ind. Hyp. for } \alpha \text{ and } \sigma(\beta)(A) \\ = & int_{\mathcal{T}}\,(\sigma(\alpha\beta)(A)) & \end{array}$$

For sum,

$$\begin{array}{lll} & \sigma(\alpha+\beta)\,(int_{\mathcal{T}}(A)) & \\ = & \sigma(\alpha)\,(int_{\mathcal{T}}(A)) \cup \sigma(\beta)\,(int_{\mathcal{T}}(A)) & \\ \subseteq & int_{\mathcal{T}}\,(\sigma(\alpha)(A)) \cup int_{\mathcal{T}}\,(\sigma(\beta)(A)) & \text{Ind. Hyp. for } \alpha, \beta \text{ and } A \\ \subseteq & int_{\mathcal{T}}\,(\sigma(\alpha)(A) \cup \sigma(\beta)(A)) & \cup \text{ property of } int_{\mathcal{T}} \\ = & int_{\mathcal{T}}\,(\sigma(\alpha+\beta)(A)) & \end{array}$$

Finally, for iteration, it is readily verified by induction on k that if $\sigma(\alpha)\,(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}\,(\sigma(\alpha)(A))$ for all $A \subseteq X$, then $\sigma(\alpha)^k\,(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}\left(\sigma(\alpha)^k(A)\right)$ for all $k \in \mathbb{N}$ and all $A \subseteq X$. Then

$$\begin{array}{lll} & \sigma(\alpha^*)\,(int_{\mathcal{T}}(A)) & \\ = & \bigcup_{k\in\mathrm{N}} \sigma(\alpha)^k\,(int_{\mathcal{T}}(A)) & \\ \subseteq & \bigcup_{k\in\mathrm{N}} int_{\mathcal{T}}\left(\sigma(\alpha)^k(A)\right) & \text{result for } \sigma(\alpha)^k \text{ and } A \\ \subseteq & int_{\mathcal{T}}\left(\bigcup_{k\in\mathrm{N}} \sigma(\alpha)^k(A)\right) & \bigcup \text{ property of } int_{\mathcal{T}} \\ = & int_{\mathcal{T}}\,(\sigma(\alpha^*)(A)) & \end{array}$$

For (ii), the ncm properties of $\pi(\alpha)$ come by duality from the nca properties of $\sigma(\alpha)$ and Proposition 5.1.4. For the continuity of $\pi(\alpha)$ on $\mathfrak{B}_{\mathcal{T}}(X)$, the hypothesis that $\mathcal{T}$ is a D-topology is needed for the α^* case of the induction, which goes as follows:

$$\begin{array}{lll} & \pi(\alpha^*)\,(int_{\mathcal{T}}(A)) & \\ = & \bigcap_{k\in\mathrm{N}} \pi(\alpha)^k\,(int_{\mathcal{T}}(A)) & \\ \subseteq & \bigcap_{k\in\mathrm{N}} int_{\mathcal{T}}\left(\pi(\alpha)^k(A)\right) & \text{result for } \pi(\alpha)^k \text{ and } A \\ = & int_{\mathcal{T}}\left(\bigcap_{k\in\mathrm{N}} int_{\mathcal{T}}\left(\pi(\alpha)^k(A)\right)\right) & \bigcap \text{ property of D-topologies} \\ \subseteq & int_{\mathcal{T}}\left(\bigcap_{k\in\mathrm{N}} \pi(\alpha)^k(A)\right) & \\ = & int_{\mathcal{T}}\,(\sigma(\alpha^*)(A)) & \end{array}$$

For (iii), the inclusion $\pi(\alpha)(A) \subseteq \sigma(\alpha)(A)$ is an easy induction on actions $\alpha \in Act(\Sigma)$. ∎

So we have established that the continuity scheme:

$$\langle\alpha\rangle\mathbf{Cont}: \;\; \langle\alpha\rangle\Box\varphi \to \Box\langle\alpha\rangle\varphi$$

is valid in all topological structures for $\mathcal{L}_{\Box}(\Phi, \Sigma)$, while the corresponding scheme for $[\alpha]$,

$$[\alpha]\mathbf{Cont}: \;\; [\alpha]\Box\varphi \to \Box[\alpha]\varphi$$

is valid in all D-topological structures. We also have that the *deontic* or *totality* scheme:

$$[\alpha]\mathbf{D}: \;\; [\alpha]\varphi \to \langle\alpha\rangle\varphi$$

is valid in all topological structures.

5.2 Kripke Semantics

Definition 5.2.1 *Let W be a non-empty set and R a reflexive and transitive binary relation on W. Let $M_R(W)$ denote the set of all R-monotone total functions $F : W \to W$.*

By Proposition 3.1.3, $M_R(W) = C_{\mathcal{T}_R}(W)$ and if $(X, \mathcal{T})$ is an D-space, then $C_{\mathcal{T}}(X) = M_{R_{\mathcal{T}}}(X)$.

Definition 5.2.2 *A (continuous)* Kripke frame *for the language $\mathcal{L}_\Box(\Phi, \Sigma)$ is a triple $\mathcal{K} = (W, R, \nu)$, where*

- *$W \neq \emptyset$ is a set of worlds;*
- *$R \subseteq W \times W$ is a reflexive and transitive binary relation on W; and*
- *$\nu : \Sigma \to M_R(W)$ is a map assigning an R-monotone total function $\nu(a) : W \to W$ to each atomic action $a \in \Sigma$.*

The map ν uniquely extends to a map $\rho = \rho_\nu : Act(\Sigma) \to \mathcal{P}(W \times W)$, assigning a binary relation $\rho(\alpha) \subseteq W \times W$ to each action $\alpha \in Act(\Sigma)$, according to the following inductive clauses, where $a \in \Sigma$ and $\alpha, \beta \in Act(\Sigma)$:

$$\begin{array}{rcl}
\rho(a) & \doteq & graph\,(\nu(a)) \\
\rho(\alpha\beta) & \doteq & \rho(\alpha) \circ \rho(\beta) \\
& = & \{(w,v) \mid (\exists u)[\,(w,u) \in \rho(\alpha) \text{ and } (u,v) \in \rho(\beta)\,]\} \\
\rho(\alpha + \beta) & \doteq & \rho(\alpha) \cup \rho(\beta) \\
\rho(\alpha^*) & \doteq & \bigcup_{k \in \mathrm{N}} \rho(\alpha)^k \;= \rho(\alpha)^{rtc}
\end{array}$$

where $\rho(\alpha)^0 \doteq \mathbf{1}_{W \times W}$ is the identity binary relation on W, $\rho(\alpha)^{k+1} \doteq \rho(\alpha) \circ \rho(\alpha)^k$, and $\rho(\alpha)^{rtc}$ is the reflexive and transitive closure of $\rho(\alpha)$.

A valuation *for a Kripke frame $\mathcal{K} = (W, R, \nu)$ is a map $\eta : W \to \mathcal{P}(\Phi)$ assigning a set of atomic propositions $\eta(w) \subseteq \Phi$ to each world $w \in W$. Each such valuation η for $\mathcal{K}$ uniquely extends to a forcing relation $\Vdash_\eta^{\mathcal{K}} = \Vdash_\eta \subseteq W \times Form(\Phi, \Sigma)$, between worlds $w \in W$ and formulas $\varphi \in Form(\Phi, \Sigma)$, according to the following inductive clauses, where $p \in \Phi$, $\varphi, \psi \in Form(\Phi, \Sigma)$ and $\alpha \in Act(\Sigma)$, for all $w \in W$:*

$$\begin{array}{rcl}
w \Vdash_\eta p & \textit{iff} & p \in \eta(w) \\
w \Vdash_\eta \neg\varphi & \textit{iff} & w \nVdash_\eta \varphi \\
w \Vdash_\eta \varphi \to \psi & \textit{iff} & w \nVdash_\eta \varphi \text{ or } w \Vdash_\eta \psi \\
w \Vdash_\eta \Box\varphi & \textit{iff} & \text{for all } v \in W, \text{ if } (w,v) \in R \text{ then } v \Vdash_\eta \varphi \\
w \Vdash_\eta \langle\alpha\rangle\varphi & \textit{iff} & \text{for some } u \in W, \; (w,u) \in \rho(\alpha) \text{ and } u \Vdash_\eta \varphi
\end{array}$$

A Kripke model *for $\mathcal{L}_\Box(\Phi, \Sigma)$ is a pair $(\mathcal{K}, \eta)$, where η is a valuation for $\mathcal{K}$.*

For formulas $\varphi \in Form(\Phi, \Sigma)$, the notions of satisfiability *and* truth *in a Kripke model $(\mathcal{K}, \eta)$,* validity *in a frame $\mathcal{K}$, and* Kripke validity, *are as in Definition 2.3.4.*

The $[\alpha]$ modality has the standard "Box" reading in the Kripke semantics:

$$w \Vdash_\eta [\alpha]\varphi \quad \text{iff} \quad \text{for all } u \in W, \text{ if } (w,u) \in \rho(\alpha) \text{ then } u \Vdash_\eta \varphi$$

The next task is to establish the duality between D-topological models and Kripke models, and with that, the duality between the unary operators $\sigma(\alpha)$ (and $\pi(\alpha)$) of topological models and the binary relations $\rho(\alpha)$ of a Kripke frame. The translations come from [JT51], Section 3, where a one-one correspondence is established between nca unary (indeed n-ary) operators on the Boolean algebra on $\mathcal{P}(X)$ and binary (or $(n+1)$-ary) relations on a set X.

Definition 5.2.3 *A topological structure $\mathfrak{T} = (X, \mathcal{T}, \nu)$ for $\mathcal{L}_\Box(\Phi, \Sigma)$ is called an* D-topological structure *iff $(X, \mathcal{T})$ is an D-space.*

Given a Kripke frame $\mathcal{K} = (W, R, \nu)$ for $\mathcal{L}_\Box(\Phi, \Sigma)$, define $\mathfrak{T}_\mathcal{K} = (W, \mathcal{T}_R, \nu)$ to be its dual I-topological structure. (By Proposition 3.1.3, $M_R(W) = C_{\mathcal{T}_R}(W)$, so the assignment map $\nu : \Sigma \to M_R(W)$ in $\mathcal{K}$ is suitable as an assignment map $\nu : \Sigma \to C_{\mathcal{T}_R}(W)$ in $\mathfrak{T}_\mathcal{K}$.)

Similarly, given an D-topological structure $\mathfrak{T} = (X, \mathcal{T}, \nu)$ for $\mathcal{L}_\Box(\Phi, \Sigma)$, define $\mathcal{K}_\mathfrak{T} = (X, R_\mathcal{T}, \nu)$ to be its dual Kripke frame.

Duality for valuations is defined as in Definition 2.4.11.

Proposition 5.2.4 Duality of Kripke frames & D-topological structures

(i) *Let $\mathcal{K} = (W, R, \nu)$ be a Kripke frame for $\mathcal{L}_\Box(\Phi, \Sigma)$, and let $\mathfrak{T}_\mathcal{K} = (W, \mathcal{T}_R, \nu)$ be its dual D-topological structure. Let $\rho = \rho_\nu : Act(\Sigma) \to \mathcal{P}(W \times W)$ be the relation map for actions in $\mathcal{K}$, and let $\sigma = \sigma_\nu : Act(\Sigma) \to (\mathcal{P}(W) \to \mathcal{P}(W))$ be the operator map for actions in $\mathfrak{T}_\mathcal{K}$. Then for all worlds $w \in W$, all $\alpha \in Act(\Sigma)$ and all $A \in \mathcal{P}(W)$,*

$$w \in \sigma(\alpha)(A) \quad \textit{iff} \quad (\exists v \in W)[(w,v) \in \rho(\alpha) \textit{ and } v \in A] \quad \text{(Op)}$$

(ii) *Let $\mathfrak{T} = (X, \mathcal{T}, \nu)$ be a D-topological structure for $\mathcal{L}_\Box(\Phi, \Sigma)$, and let $\mathcal{K}_\mathfrak{T} = (X, R_\mathcal{T}, \nu)$ be its dual Kripke frame. Let $\sigma = \sigma_\nu : Act(\Sigma) \to (\mathcal{P}(X) \to \mathcal{P}(X))$ be the operator map for actions in $\mathfrak{T}$, and let $\rho = \rho_\nu : Act(\Sigma) \to \mathcal{P}(X \times X)$ be the relation map for actions in $\mathcal{K}_\mathfrak{T}$. Then for all states $x, y \in X$, and all $\alpha \in Act(\Sigma)$,*

$$(x,y) \in \rho(\alpha) \quad \textit{iff} \quad x \in \sigma(\alpha)(\{y\}) \qquad \text{(Rel)}$$

Proof. We prove (i) by induction on actions $\alpha \in Act(\Sigma)$; (ii) is then a consequence of [JT51], Theorem 3.3, which uses the equivalences (Op) and (Rel) to establish a one-one correspondence between nca unary operators on $\mathcal{P}(X)$ and binary relations on X.

For an atomic action $a \in \Sigma$, let $\nu(a) = F \in M_R(W) = C_{\mathcal{T}_R}(W)$. Then $\sigma(a) = F^{-1}$ and $\rho(a) = graph\,(F)$. Hence

$$\begin{array}{ll} & w \in \sigma(a)\,(A) \\ \Leftrightarrow & w \in F^{-1}(A) \\ \Leftrightarrow & F(w) \in A \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in graph\,(F) \text{ and } v \in A\,] \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(a) \text{ and } v \in A\,] \end{array}$$

For composition, assume the result (Op) holds for $\alpha, \beta \in Act(\Sigma)$, for all $A \in \mathcal{P}(W)$ and $w \in W$. Then

$$\begin{array}{lll} & w \in \sigma(\alpha\beta)\,(A) & \\ \Leftrightarrow & w \in (\sigma(\alpha) \circ \sigma(\beta))\,(A) & (1) \\ \Leftrightarrow & w \in \sigma(\alpha)\,(\sigma(\beta)\,(A)) & \\ \Leftrightarrow & (\exists u \in W)[\,(w,u) \in \rho(\alpha) \text{ and } u \in \sigma(\beta)\,(A)\,] & (2) \\ \Leftrightarrow & (\exists u \in W)[\,(w,u) \in \rho(\alpha) \text{ and } (\exists v \in W)[\,(u,v) \in \rho(\beta) \text{ and } v \in A]] & (3) \\ \Leftrightarrow & (\exists v \in W)\,(\exists u \in W)[\,(w,u) \in \rho(\alpha) \text{ and } (u,v) \in \rho(\beta) \text{ and } v \in A\,] & \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(\alpha) \circ \rho(\beta) \text{ and } v \in A\,] & \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(\alpha\beta) \text{ and } v \in A\,] & (4) \end{array}$$

where (1) is by definition of $\sigma(\alpha\beta)$; (2) is (Op) for α, $B = \sigma(\beta)\,(A)$ and w; (3) is (Op) for β, A and u; and (4) is by definition of $\rho(\alpha\beta)$.

For sum, assume (Op) holds for $\alpha, \beta \in Act(\Sigma)$, for all $A \in \mathcal{P}(W)$ and $w \in W$.

$$\begin{array}{lll} & w \in \sigma(\alpha + \beta)\,(A) & \\ \Leftrightarrow & w \in \sigma(\alpha)\,(A) \cup \sigma(\beta)(A) & (1) \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(\alpha) \text{ and } v \in A\,] \text{ or} & \\ & (\exists v \in W)[\,(w,v) \in \rho(\beta) \text{ and } v \in A\,] & (2) \\ \Leftrightarrow & (\exists v \in W)[\,[(w,v) \in \rho(\alpha) \text{ or } (w,v) \in \rho(\beta)] \text{ and } v \in A\,] & \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(\alpha + \beta) \text{ and } v \in A\,] & (3) \end{array}$$

where (1) is by definition of $\sigma(\alpha + \beta)$; (2) is (Op) for α and β, A and w; and (3) is by definition of $\rho(\alpha + \beta)$.

For iteration, assume (Op) holds for $\alpha \in Act(\Sigma)$, for all $A \in \mathcal{P}(W)$ and $w \in W$. It is readily verified, by induction on k, that (Op) holds for $\sigma(\alpha)^k$ for all $k \in \mathbb{N}$ and

all $A \in \mathcal{P}(W)$ and $w \in W$. Then

$$\begin{array}{lll} & w \in \sigma(\alpha^*)(A) & \\ \Leftrightarrow & w \in \bigcup_{k\in\mathbb{N}} \left(\sigma(\alpha)^k\right)(A) & (1) \\ \Leftrightarrow & (\exists k \in \mathbb{N})[\, w \in \left(\sigma(\alpha)^k\right)(A)\,] & \\ \Leftrightarrow & (\exists k \in \mathbb{N})(\exists v \in W)[\,(w,v) \in \rho(\alpha)^k \text{ and } v \in A\,] & (2) \\ \Leftrightarrow & (\exists v \in W)(\exists k \in \mathbb{N})[\,(w,v) \in \rho(\alpha)^k \text{ and } v \in A\,] & \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \bigcup_{k\in\mathbb{N}} \rho(\alpha)^k \text{ and } v \in A\,] & \\ \Leftrightarrow & (\exists v \in W)[\,(w,v) \in \rho(\alpha^*) \text{ and } v \in A\,] & (3) \end{array}$$

where (1) is by definition of $\sigma(\alpha^*)$; (2) is (Op) for α^k, A and w; and (3) is by definition of $\rho(\alpha^*)$. ■

Note that since $\sigma(\alpha)$ is completely additive,

$$\sigma(\alpha)(A) = \bigcup_{y\in A} \sigma(\alpha)(\{y\})$$

In establishing the duality transformation between Kripke models and D-topological models, the equivalences (Op) and (Rel) are exactly what is needed for the $\langle\alpha\rangle$ case of the induction on formulas.

Proposition 5.2.5 Duality of Kripke & D-topological models

(i) *Let $(\mathcal{K},\eta)$ be a Kripke model for $\mathcal{L}_\Box(\Phi,\Sigma)$, and let $(\mathfrak{T}_\mathcal{K},\xi_\eta)$ be its dual D-topological model. Let $\rho : Act(\Sigma) \to \mathcal{P}(W \times W)$ be the relation map for actions in $\mathcal{K}$, and let $\sigma : Act(\Sigma) \to (\mathcal{P}(W) \to \mathcal{P}(W))$ be the operator map for actions in $\mathfrak{T}_\mathcal{K}$. Then for all worlds $w \in W$, all $\alpha \in Act(\Sigma)$ and all $\varphi \in Form(\Phi,\Sigma)$,*

$$w \in \|\varphi\|_{\xi_\eta} \quad \textit{iff} \quad w \Vdash_\eta \varphi$$

Hence

$$(\mathfrak{T}_\mathcal{K},\xi_\eta) \models \varphi \quad \textit{iff} \quad (\mathcal{K},\eta) \Vdash \varphi$$

(ii) *Let $(\mathfrak{T},\xi)$ be a D-topological model for $\mathcal{L}_\Box(\Phi,\Sigma)$, and let $(\mathcal{K}_\mathfrak{T},\eta_\xi)$ be its dual Kripke model. Let $\sigma : Act(\Sigma) \to (\mathcal{P}(X) \to \mathcal{P}(X))$ be the operator map for actions in $\mathfrak{T}$, and $\rho : Act(\Sigma) \to \mathcal{P}(X \times X)$ be the relation map for actions in $\mathcal{K}_\mathfrak{T}$. Then for all states $x \in X$, all $\alpha \in Act(\Sigma)$, and all $\varphi \in Form(\Phi,\Sigma)$,*

$$x \Vdash_{\eta_\xi} \varphi \quad \textit{iff} \quad x \in \|\varphi\|_\xi$$

Hence

$$(\mathcal{K}_\mathfrak{T},\eta_\xi) \Vdash \varphi \quad \textit{iff} \quad (\mathfrak{T},\xi) \models \varphi$$

Proof. For (i), proceed by induction on the sub-formula ordering, extending the proof of Proposition 2.4.12, with the new $\langle\alpha\rangle\varphi$ case appealing to the (Op) equivalence; similarly for (ii) using the (Op) equivalence. ■

Corollary 5.2.6 *For all formulas $\varphi \in Form(\Phi, \Sigma)$,*

$$\begin{array}{rl} & \mathfrak{T} \models \varphi \ \textit{ for all D-topological structures } \mathfrak{T} \ \textit{ for } \mathcal{L}_\square(\Phi,\Sigma) \\ \textit{iff} & \mathcal{K} \Vdash \varphi \ \textit{ for all Kripke frames } \mathcal{K} \textit{ for } \mathcal{L}_\square(\Phi,\Sigma) \end{array}$$

Corollary 5.2.7 *For all formulas $\varphi \in Form(\Phi, \Sigma)$,*

$$\begin{array}{rl} & \mathfrak{T} \models \varphi \ \textit{ for all } T_0 \textit{ D-topological structures } \mathfrak{T} \ \textit{ for } \mathcal{L}_\square(\Phi,\Sigma) \\ \textit{iff} & \mathcal{K} \Vdash \varphi \ \textit{ for all partially ordered Kripke frames } \mathcal{K} \textit{ for } \mathcal{L}_\square(\Phi,\Sigma) \end{array}$$

Corollary 5.2.8 *For all Kripke frames $\mathcal{K}$ for $\mathcal{L}_\square(\Phi,\Sigma)$, and all $\varphi \in Form(\Phi, \Sigma)$,*

$$\begin{array}{l} \mathcal{K} \Vdash \langle\alpha\rangle\square\varphi \to \square\langle\alpha\rangle\varphi \\ \mathcal{K} \Vdash [\alpha]\square\varphi \to \square[\alpha]\varphi \\ \mathcal{K} \Vdash [\alpha]\varphi \to \langle\alpha\rangle\varphi \end{array}$$

5.3 Expressivity of TPDL

In this section, we briefly give examples of the sorts of properties that can be expressed in the language of **TPDL**. It is by no means an exhaustive study of the "expressive power" — that will have to await a further investigation — but rather a few suggestive examples.

Various *reachability* properties can be expressed. Fix a topological model $(\mathfrak{T}, \xi)$, with $\mathfrak{T} = (X, \mathcal{T}, \nu)$. Then we have:

$$\begin{array}{lll} (\mathfrak{T},\xi) \models \psi \to \langle\alpha^*\rangle\varphi & \text{iff} & \text{from } \|\psi\|_\xi \text{, some iteration of } \alpha \text{ leads to } \|\varphi\|_\xi \\ (\mathfrak{T},\xi) \models \psi \to [\alpha^*]\varphi & \text{iff} & \text{from } \|\psi\|_\xi \text{, every iteration of } \alpha \text{ leads to } \|\varphi\|_\xi \end{array}$$

Similarly, the formulas

$$\psi \to \langle\alpha^*\rangle\square\varphi \quad \text{and} \quad \psi \to [\alpha^*]\square\varphi$$

express a *topological reachability* properties, requiring $int_\mathcal{T}\left(\|\varphi\|_\xi\right)$ to be reachable by some or all iterations of α from $\|\psi\|_\xi$. Thinking if $int_\mathcal{T}$ as allowing a "margin of error" or a "robustness" in the presence of imprecision, many useful properties will

of this form. Moreover, continuity says that we can safely "push" operators $\langle\alpha\rangle$ and, in D-spaces, $[\alpha]$, "through $\Box$'s". The continuous analogs of the Hoare composition rules:

$$\frac{\psi \to \langle\alpha\rangle\Box\chi \quad \chi \to \langle\beta\rangle\Box\varphi}{\psi \to \langle\alpha\beta\rangle\Box\varphi} \text{ and } \frac{\psi \to [\alpha]\Box\chi \quad \chi \to [\beta]\Box\varphi}{\psi \to [\alpha\beta]\Box\varphi}$$

are truth-preserving in all topological structures, and all D-topological structures, respectively.

One can think of the word

$$u = a_1 \cdots a_n \in \Sigma^*$$

as a *basic control script*: a finite sequence of atomic actions. Suppose $\nu(a_j) = f_j$ for $1 \leq j \leq n$, and let $g = f_n \circ \cdots \circ f_1$. Then g is a continuous function, and extending ν to words $u \in \Sigma^*$, we have $\nu(u) = g$ and $\sigma(u) = g^{-1}$. Then

$$x \in \sigma(a_1 \cdots a_n)(A) \quad \text{iff} \quad g(x) = (f_n \circ \cdots \circ f_1)(x) \in A$$

Hence

$$x \in \|\langle u\rangle\varphi\|_\xi \ \text{ iff } \ x \in \|[u]\varphi\|_\xi \ \text{ iff } \ g(x) \in \|\varphi\|_\xi$$

and

$$\begin{array}{lll} x \in \|\langle u^*\rangle\varphi\|_\xi & \text{iff} & \text{for some } k \in \mathbb{N},\ g^k(x) \in \|\varphi\|_\xi \\ x \in \|[u^*]\varphi\|_\xi & \text{iff} & \text{for all } k \in \mathbb{N},\ g^k(x) \in \|\varphi\|_\xi \end{array}$$

The formula $\langle u^*\rangle\varphi \to \varphi$, whose converse is valid, defines the property of *weak closure under* g, since:

$$\begin{array}{ll} & (\mathfrak{T},\xi) \models \langle u^*\rangle\varphi \to \varphi \\ \Leftrightarrow & (\mathfrak{T},\xi) \models \langle u^*\rangle\varphi \leftrightarrow \varphi \\ \Leftrightarrow & \text{for all } x \in X, \quad x \in \|\varphi\|_\xi \ \text{ iff } \ \text{for some } k \in \mathbb{N},\ g^k(x) \in \|\varphi\|_\xi \end{array}$$

i.e. every $x \in \|\varphi\|_\xi$ eventually returns to $\|\varphi\|_\xi$ by some iteration of g. Equivalently, $\|\varphi\|_\xi$ is the *greatest fixed point* of $\sigma(u) = g^{-1}$.

Similarly, the formula $\varphi \to [u^*]\varphi$ defines the property of *strong closure under* g, since:

$$\begin{array}{ll} & (\mathfrak{T},\xi) \models \varphi \to [u^*]\varphi \\ \Leftrightarrow & (\mathfrak{T},\xi) \models [u^*]\varphi \leftrightarrow \varphi \\ \Leftrightarrow & \text{for all } x \in X, \quad x \in \|\varphi\|_\xi \ \text{ iff } \ \text{for all } k \in \mathbb{N},\ g^k(x) \in \|\varphi\|_\xi \end{array}$$

i.e. every $x \in \|\varphi\|_\xi$ remains in $\|\varphi\|_\xi$ under all iterations of g. Equivalently, $\|\varphi\|_\xi$ is the *least fixed point* of $\sigma(u) = g^{-1}$.

5.4 Hilbert-style Proof System

We overlay the axiomatization for **PDL** given in [Ha84], §2.2,2.5, on top of **S4C**, with atomic actions as continuous total functions. The axiomatization given here is not a minimal axiomatization; for example, we probably only need the $\leftarrow$ direction of the $\langle\alpha^*\rangle$ scheme, and the continuity scheme for arbitrary actions $\langle\alpha\rangle$**Cont** should be derivable from corresponding scheme for atomic actions. It should rather be thought of as a useful reference list.

Definition 5.4.1 *The Hilbert-style proof system for the logic* **TPDL** *has the following axiom schemes, for atomic actions $a \in \Sigma$, $\varphi, \psi \in Form(\Phi, \Sigma)$ and $\alpha, \beta \in Act(\Sigma)$:*

CP :	*axioms of classical propositional logic in $Form(\Phi, \Sigma)$*
$\Box$**K** :	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box$**T** :	$\Box\varphi \to \varphi$
$\Box$**4** :	$\Box\varphi \to \Box\Box\varphi$
$[a]$**F** :	$[a]\varphi \leftrightarrow \langle a\rangle\varphi$
$[\alpha]$**K** :	$[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
$[\alpha]$**D** :	$[\alpha]\varphi \to \langle\alpha\rangle\varphi$
$\langle\alpha\rangle$**Cont** :	$\langle\alpha\rangle\Box\varphi \to \Box\langle\alpha\rangle\varphi$
$\langle\alpha\rangle\vee$:	$\langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$
$\langle\alpha\beta\rangle$:	$\langle\alpha\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$
$\langle\alpha+\beta\rangle$:	$\langle\alpha+\beta\rangle\varphi \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi)$
$\langle\alpha^*\rangle$:	$\langle\alpha^*\rangle\varphi \leftrightarrow (\varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi)$
$\langle\alpha^*\rangle$**Ind** :	$\langle\alpha^*\rangle\varphi \to (\varphi \vee \langle\alpha^*\rangle(\neg\varphi \wedge \langle\alpha\rangle\varphi))$

and the inference rules:

modus ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

$\Box$*-necessitation:* $\dfrac{\varphi}{\Box\varphi}$

$[\alpha]$*-necessitation:* $\dfrac{\varphi}{[\alpha]\varphi}$

We write

$$\mathbf{TPDL} \vdash_H \varphi$$

or say φ is $\mathbf{TPDL}_H$ *provable, if the formula $\varphi \in Form(\Phi, \Sigma)$ has an* **TPDL** *Hilbert-style derivation.*

The following formulas are $\mathbf{TPDL}_H$ provable, for any for atomic actions $a_1, ..., a_n \in \Sigma$, $\varphi, \psi \in Form(\Phi, \Sigma)$ and $\alpha, \beta \in Act(\Sigma)$, and $k \in \mathbb{N}$:

$$\begin{array}{ll}
[\alpha]\mathbf{Cont}: & \Diamond[\alpha]\varphi \to [\alpha]\Diamond\varphi \\
[\alpha]\wedge: & [\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi) \\
[\alpha]\vee: & ([\alpha]\varphi \vee [\alpha]\psi) \to [\alpha](\varphi \vee \psi) \\
[\alpha\beta]: & [\alpha\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi \\
[\alpha+\beta]: & [\alpha+\beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi) \\
[\alpha^*]: & [\alpha^*]\varphi \leftrightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi) \\
[\alpha^*]\mathbf{Ind}: & (\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi)) \to [\alpha^*]\varphi \\
[\alpha^*][\alpha^k]: & [\alpha^*]\varphi \to [\alpha^k]\varphi \\
[\alpha^*]\mathbf{T}: & [\alpha^*]\varphi \to \varphi \\
[\alpha^*]\mathbf{4}: & [\alpha^*][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi
\end{array}$$

$$\begin{array}{ll}
\langle\alpha\rangle\wedge: & \langle\alpha\rangle(\varphi \wedge \psi) \to (\langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi) \\
\langle\alpha^*\rangle\langle\alpha^k\rangle: & \langle\alpha^k\rangle\varphi \to \langle\alpha^*\rangle\varphi \\
\langle\alpha^*\rangle\mathbf{T}: & \varphi \to \langle\alpha^*\rangle\varphi \\
\langle\alpha^*\rangle\mathbf{4}: & \langle\alpha^*\rangle\langle\alpha^*\rangle\varphi \leftrightarrow \langle\alpha^*\rangle\varphi
\end{array}$$

$$\begin{array}{ll}
[a_1 \cdots a_n]\mathbf{F}: & [a_1 \cdots a_n]\varphi \leftrightarrow \langle a_1 \cdots a_n\rangle\varphi \\
\langle a_1 \cdots a_n\rangle\neg: & \neg\langle a_1 \cdots a_n\rangle\varphi \leftrightarrow \langle a_1 \cdots a_n\rangle\neg\varphi \\
\langle a_1 \cdots a_n\rangle\to: & \langle a_1 \cdots a_n\rangle(\varphi \to \psi) \leftrightarrow (\langle a_1 \cdots a_n\rangle\varphi \to \langle a_1 \cdots a_n\rangle\psi) \\
\langle a_1 \cdots a_n\rangle\wedge: & \langle a_1 \cdots a_n\rangle(\varphi \wedge \psi) \leftrightarrow (\langle a_1 \cdots a_n\rangle\varphi \wedge \langle a_1 \cdots a_n\rangle\psi) \\
\langle a_1 \cdots a_n\rangle\vee: & \langle a_1 \cdots a_n\rangle(\varphi \vee \psi) \leftrightarrow (\langle a_1 \cdots a_n\rangle\varphi \vee \langle a_1 \cdots a_n\rangle\psi) \\
\langle a_1 \cdots a_n\rangle\top: & \langle a_1 \cdots a_n\rangle\top \\
\langle a_1 \cdots a_n\rangle\bot: & \langle a_1 \cdots a_n\rangle\bot \leftrightarrow \bot
\end{array}$$

This last block of derivable formulas asserts that compositions $(a_1 \cdots a_n)$ of atomic actions behave just like a single atomic action a: they are functional and continuous, and commute with all the Boolean operations.

Proposition 5.4.2 Soundness of **TPDL** Hilbert-style proof system
For all formulas $\varphi \in Form(\Phi, \Sigma)$, if $\mathbf{TPDL} \vdash_H \varphi$ *then* $\mathfrak{T} \models \varphi$ *for all topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$*, and* $\mathcal{K} \Vdash \varphi$ *for all Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$.

Proof. An easy extension of Propositions 2.2.2, 2.3.5 and 3.2.2. The topological and Kripke validity of the $\langle\alpha\rangle$**Cont** and $[\alpha]$**D** axiom comes Proposition 5.1.7 and Corollary 5.2.8, and the validity of the **PDL** axioms for compound actions is a straightforward exercise. ■

Completeness proofs for the axiomatization of **PDL** (e.g. [KP81], [Seg82], [KT90]) can be easily adapted to **TPDL**. As is for all dynamic logics, the infinitary nature of the iteration operation is the core complication which prevents the "cheap" maximal consistent sets construction from going through quite so smoothly.

In virtue of the equation $\rho(\alpha^*) = \rho(\alpha)^{rtc}$ among the inductive clauses extending the relation map ρ from atomic actions to all actions in Definition 5.2.2, our Kripke models are by definition *standard* in the sense of [Koz80]: $\rho(\alpha^*)$ is defined to be the reflexive and transitive closure of $\rho(\alpha)$.

Definition 5.4.3 *A* non-standard *Kripke model for $\mathcal{L}_\Box(\Phi, \Sigma)$ is a quadruple*

$$\mathcal{M} = (W, R, \{R(\alpha)\}_{\alpha \in Act(\Sigma)}, \eta)$$

where

- $W \neq \emptyset$;
- $R \subseteq W \times W$ *is reflexive and transitive;*
- $\eta : W \to P(\Phi)$ *is a valuation; and*
- *for each* $\alpha \in Act(\Sigma)$, $R(\alpha) \subseteq W \times W$ *is a relation satisfying:*
  - (i) *for each atomic action* $a \in \Sigma$, $R(a)$ *is total, functional, and R-monotone;*
  - (ii) $R(\alpha\beta) = R(\alpha) \circ R(\beta)$;
  - (iii) $R(\alpha + \beta) = R(\alpha) \cup R(\beta)$; *and*
  - (iv) $R(\alpha^*)$ *is a reflexive and transitive relation containing* $R(\alpha)$ *(and hence containing* $R(\alpha)^{rtc}$*) and also satisfying the* $\langle\alpha^*\rangle$**Ind** *induction axiom, which means: for all* $w \in W$ *and* $A \subseteq W$, *if for some* $v \in W$, $(w, v) \in R(\alpha^*)$ *and* $v \in A$, *then either* $w \in A$ *or there exists* $u, z \in W$ *such that* $(w, u) \in R(\alpha^*)$, $(u, z) \in R(\alpha)$, $u \notin A$ *and* $z \in A$.

Now define $\mathcal{M}_0 = (W_0, R_0, \{R_0(\alpha)\}_{\alpha \in Act(\Sigma)}, \eta_0)$ by:

$$\begin{gathered}
W_0 \doteq \{U \subseteq Form(\Phi, \Sigma) \mid U \text{ is maximal } \mathbf{TPDL}\text{-consistent}\} \\
(U, V) \in R_0 \text{ iff } (\forall \varphi \in Form(\Phi, \Sigma))[\, \Box\varphi \in U \Rightarrow \varphi \in V\,] \\
(U, V) \in R_0(\alpha) \text{ iff } (\forall \varphi \in Form(\Phi, \Sigma))[\, \varphi \in V \Rightarrow \langle\alpha\rangle\varphi \in U\,] \\
\eta_0(U) \doteq \{p \in \Phi \mid p \in U\}
\end{gathered}$$

From the proofs of Propositions 2.3.7 and 3.2.3, R_0 is a reflexive and transitive binary relation on W_0, and for each atomic action $a \in \Sigma$, the relation $R_0(a)$ defines the total function:

$$(F_0(a))(U) = \{\varphi \in Form(\Phi, \Sigma) \mid \langle a \rangle \varphi \in U\}$$

which "peels-off" one $\langle a \rangle$; moreover, each $F_0(a)$ is R_0-monotone.

Lemma 5.4.4 *The structure $\mathcal{M}_0 = (W_0, R_0, \{R_0(\alpha)\}_{\alpha \in Act(\Sigma)}, \eta_0)$ is a non-standard Kripke model for $\mathcal{L}_\Box(\Phi, \Sigma)$, and for all $\varphi \in Form(\Phi, \Sigma)$ and $U \in W_0$,*

$$U \Vdash_{\eta_0} \varphi \quad \textit{iff} \quad \varphi \in U$$

Proof. For the first part, it suffices to show that the relations $\{R_0(\alpha)\}_{\alpha \in Act(\Sigma)}$ satisfy conditions (ii), (iii), and (iv). The reasoning is identical to that for **PDL**; see, for example, [Seg82], §4. The "Truth Lemma" is a straightforward induction on formulas. ■

The failure of the converse of the condition:

$$R(\alpha)^{rtc} \subseteq R(\alpha^*)$$

is due to the failure of *compactness*: every finite subset of

$$C = \{\langle \alpha^* \rangle p\} \cup \{\neg \langle \alpha^k \rangle p \mid k \in \mathbb{N}\}$$

is satisfiable (in a standard Kripke model), but C is not satisfiable. To produce a *standard* Kripke model in which a $\mathbf{TPDL}_H$ non-provable formula is falsified (or in which a **TPDL**-consistent formula is satisfied), we continue to follow the pattern of **PDL** completeness proofs by taking a filtration (or quotient) through the Fischer-Ladner closure. The Fischer-Ladner order on formulas, which extends the subformula ordering by having $\psi \prec \varphi$ whenever ψ is "relevant" to the semantics of φ, is also required for the proof of completeness for tableaux in Section 5.7 below.

Definition 5.4.5 [FL79], [KT90]. *Let $\prec$ be the smallest transitive binary relation on the set $Form(\Phi, \Sigma)$ of formulas of $\mathcal{L}_\Box(\Phi, \Sigma)$ satisfying the following inequalities, for all $\varphi, \psi \in Form(\Phi, \Sigma)$ and $\alpha, \beta \in Act(\Sigma)$:*

$$\begin{array}{ll}
(\neg): & \varphi \prec \neg\varphi \\
(\rightarrow): & \varphi \prec \varphi \rightarrow \psi \;\; \textit{and} \;\; \psi \prec \varphi \rightarrow \psi \\
(\Box): & \varphi \prec \Box\varphi \\
(\alpha): & \varphi \prec \langle \alpha \rangle \varphi \\
(\alpha\beta): & \langle \alpha \rangle \langle \beta \rangle \varphi \prec \langle \alpha\beta \rangle \varphi \;\; \textit{and} \;\; \langle \beta \rangle \varphi \prec \langle \alpha\beta \rangle \varphi \\
(\alpha + \beta): & \langle \alpha \rangle \varphi \prec \langle \alpha + \beta \rangle \varphi \;\; \textit{and} \;\; \langle \beta \rangle \varphi \prec \langle \alpha + \beta \rangle \varphi \\
(\alpha^*): & \langle \alpha \rangle \langle \alpha^* \rangle \varphi \prec \langle \alpha^* \rangle \varphi
\end{array}$$

The relation $\prec$ *is called the* Fischer-Ladner order. *Define, for* $\varphi, \psi \in Form(\Phi, \Sigma)$,

$$\psi \preccurlyeq \varphi \quad \textit{iff} \quad \psi \prec \varphi \quad \textit{or} \quad \psi = \varphi$$

Then for any formula $\varphi \in Form(\Phi, \Sigma)$, *define the* Fischer-Ladner closure *of* φ, *denoted* $FL(\varphi)$, *to be the set of formulas:*

$$FL(\varphi) \doteq \{\psi \in \Phi \mid \psi \preccurlyeq \varphi\}$$

It is immediate that $SF(\varphi) \subseteq FL(\varphi)$, where $SF(\varphi)$ is the set of subformulas of φ. The crucial property is that $\prec$ is well-founded. To see this, let $|\alpha|$ and $|\varphi|$ denote the lengths of $\alpha \in Act(\Sigma)$ and $\varphi \in Form(\Phi, \Sigma)$, respectively, considered as strings over the alphabet:

$$\Sigma \cup \Phi \cup \{\cdot, +, ^*, (,)\} \cup \{\neg, \rightarrow, \Box, \langle, \rangle\}$$

where "$\cdot$" is used to denote composition. Then $\langle\beta\rangle\psi \prec \langle\alpha\rangle\varphi$ implies $|\beta| < |\alpha|$. Since only $\langle\alpha\rangle$ formulas have $\prec$-predecessors that are not subformulas, it follows that there can be no infinite descending $\prec$-chains, and every descending $\prec$-chain ends with an atomic proposition $p \in \Phi$. A straightforward induction with respect to $\prec$ establishes that:

$$|FL(\varphi)| \leq |\varphi|$$

for all $\varphi \in Form(\Phi, \Sigma)$.

Let $\mathcal{M}_0 = (W_0, R_0, \{R_0(\alpha)\}_{\alpha \in Act(\Sigma)}, \eta_0)$ be the non-standard Kripke model for $\mathcal{L}_\Box(\Phi, \Sigma)$, as above, and fix $\varphi \in Form(\Phi, \Sigma)$. Define an equivalence relation $\equiv_\varphi$ on W_0 by:

$$U \equiv_\varphi V \quad \text{iff} \quad U \cap FL(\varphi) = V \cap FL(\varphi)$$

For each $U \in W_0$, let $\bar{U} = \{V \in W_0 \mid U \equiv_\varphi V\}$ denote the $\equiv_\varphi$-equivalence class of U, and let W_φ denote the set of all such equivalence classes. Define a (standard) Kripke frame $\mathcal{K}_\varphi = (W_\varphi, R_\varphi, \nu_\varphi)$ to be the *minimal quotient* (Lemma 3.3.2, extended to Kripke frames for $\mathcal{L}_\Box(\Phi, \Sigma)$) of the frame of $\mathcal{M}_0$ under the surjective map $h : W_0 \rightarrow W_\varphi$ given by $h(U) = \bar{U}$; i.e. for all $U, V \in W_0$,

$$h(U) = h(V) \quad \text{iff} \quad U \cap FL(\varphi) = V \cap FL(\varphi)$$

and

$$(h(U), h(V)) \in R_\varphi \quad \text{iff} \quad (U, V) \in R_0$$

and for each atomic action $a \in \Sigma$, where $graph(F_0(a)) = R_0(a)$ in $\mathcal{M}_0$,

$$(\nu_\varphi(a))(h(U)) = h((F_0(a))(U))$$

i.e. $\nu_\varphi(a) \circ h = h \circ F_0(a)$. Define a valuation $\eta_\varphi : W_\varphi \to P(\Phi)$ for $\mathcal{K}_\varphi$ by:

$$\eta_\varphi(h(U)) \doteq \{p \in \Phi \mid p \in U\}$$

The Kripke model $(\mathcal{K}_\varphi, \eta_\varphi)$ is commonly called the *filtration* of $\mathcal{M}_0$ through $FL(\varphi)$. Note that $\mathcal{K}_\varphi$ has at most $2^{|\varphi|}$ worlds (states).

Lemma 5.4.6 Filtration Lemma.

(a) *For all $\psi \in FL(\varphi)$ and all $U \in W_0$,*

$$h(U) \Vdash_{\eta_\varphi} \psi \quad \textit{iff} \quad U \Vdash_{\eta_0} \psi$$

(b) *For all $\langle\alpha\rangle\psi \in FL(\varphi)$ and all $U, V \in W_0$,*

$$\textit{if} \quad (U, V) \in R_0(\alpha) \quad \textit{then} \quad (h(U), h(V)) \in \rho(\alpha)$$

and

$$\textit{if} \quad (h(U), h(V)) \in \rho(\alpha) \textit{ and } \psi \in V \quad \textit{then} \quad \langle\alpha\rangle\psi \in U$$

where ρ is the induced relation map for $(\mathcal{K}_\varphi, \eta_\varphi)$.

Proof. By induction on the Fischer-Ladner order $\prec$. See [KT90], [Seg82], Lemmas 6.3A and 6.3B. ■

Proposition 5.4.7 Kripke Completeness of **TPDL** Hilbert-style proof system
For each $\varphi \in Form(\Phi, \Sigma)$, if **TPDL** $\nvdash_H \varphi$, *then there exists a (finite) Kripke model $(\mathcal{K}_\varphi, \eta_\varphi)$ for $\mathcal{L}_\Box(\Phi, \Sigma)$ such that $(\mathcal{K}_\varphi, \eta_\varphi) \nVdash \varphi$.*

Proof. Fix $\varphi \in Form(\Phi, \Sigma)$ and suppose **TPDL** $\nvdash_H \varphi$. Then $\{\neg\varphi\}$ is **TPDL**-consistent, so there exists a maximal **TPDL**-consistent set U such that $\neg\varphi \in U$; equivalently, $\varphi \notin U$. Hence in the non-standard model $\mathcal{M}_0$, we have $U \nVdash_{\eta_0} \varphi$, by Lemma 5.4.4. Let $(\mathcal{K}_\varphi, \eta_\varphi)$ be the filtration of $\mathcal{M}_0$ through $FL(\varphi)$. Then by Lemma 5.4.6, we have $h(U) \nVdash_{\eta_\varphi} \varphi$. Hence $(\mathcal{K}_\varphi, \eta_\varphi) \nVdash \varphi$, as required. ■

5.5 Tableaux Proof System

The tableaux proof system developed for the logics **S4F** and **S4C** readily extends to **TPDL.** The tableaux rules for complex action modalities $\langle\alpha\rangle$ reflect the corresponding axioms in the Hilbert-style system.

Definition 5.5.1 *Let* $\mathbf{W} = \{\mathbf{w}_i \mid i \in \mathbb{N}\}$ *be a countable set of* world symbols *and let* $\mathcal{F} = \{\mathbf{F}_j \mid j \in \mathbb{N}\}$ *be a countable family of unary function symbols. Let* $\mathbf{W}(\mathcal{F})$ *be the set of terms, called* world terms, *generated from* $\mathbf{W}$ *under* $\mathcal{F}$. *So every world term* $t \in \mathbf{W}(\mathcal{F})$ *is either a world symbol* $\mathbf{w}_i \in \mathbf{W}$, *or else of the form* $(\mathbf{F}_{j_n} \circ \cdots \circ \mathbf{F}_{j_1})(\mathbf{w}_i)$ *for some function symbols* $\mathbf{F}_{j_1}, ..., \mathbf{F}_{j_n} \in \mathcal{F}$.

To simplify notation, let $\mathbb{N}^*$ *denote the set of all finite strings (sequences) over* $\mathbb{N}$, *with the empty string* $\lambda \in \mathbb{N}^*$. *For each* $i \in \mathbb{N}$ *and* $\sigma \in \mathbb{N}^*$, *define the term* $\mathbf{F}_\sigma(\mathbf{w}_i)$ *by induction on strings as follows:*

$$\begin{aligned} \mathbf{F}_\lambda(\mathbf{w}_i) &\doteq \mathbf{w}_i \\ \mathbf{F}_{\sigma\hat{\ }j}(\mathbf{w}_i) &\doteq \mathbf{F}_j(\mathbf{F}_\sigma(\mathbf{w}_i)) \end{aligned}$$

where $\sigma\hat{\ }j$ *is the result of adjoining* $j \in \mathbb{N}$ *to the string* σ.

It is immediate that $\mathbf{W}(\mathcal{F})$ is in one-one correspondence with the set $\mathbb{N} \times \mathbb{N}^*$ under $t \rightleftarrows (i, \sigma)$ iff $t = \mathbf{F}_\sigma(\mathbf{w}_i)$. We also assume we have a fixed enumeration $\Sigma = \{a_j \mid j \in \mathbb{N}\}$ of atomic actions of the language $\mathcal{L}_\Box(\Phi, \Sigma)$. In the canonical term frame, the atomic action a_j will be interpreted by the term constructor function $t \mapsto \mathbf{F}_j(t)$, or in the string notation, $\mathbf{F}_\sigma(\mathbf{w}_i) \mapsto \mathbf{F}_{\sigma\hat{\ }j}(\mathbf{w}_i)$.

Definition 5.5.2 *For each formula* $\varphi \in \mathcal{L}_\Box(\Phi, \Sigma)$, *define* Σ_φ *to be the set of indices of atomic actions appearing in* φ:

$$\Sigma_\varphi \doteq \{j \in \mathbb{N} \mid \langle a_j\rangle\psi \in FL(\varphi)\}$$

Let Σ^*_φ *denote the set of all finite sequences over* Σ_φ.

If $t = \mathbf{F}_\sigma(\mathbf{w}_i)$ *and* $\sigma \in \Sigma^*_\varphi$, *then* t *is said to be* relevant *to any signed forcing assertion* $T[\ \mathbf{w}_i \Vdash \varphi\ ]$ *or* $F[\ \mathbf{w}_i \Vdash \varphi\ ]$ *which has* $\mathbf{w}_i$ *as its subject and* φ *as its object.*

Definition 5.5.3 *The class of* atomic tableaux *includes the labeled binary trees (T-AP), (F-AP), (T$\neg$), (F$\neg$), (T $\rightarrow$),(F $\rightarrow$), (T$\Box$), and (F$\Box$), of Definition 4.1.3, with atomic formulas* $p \in \Phi = AP$, *formulas* $\varphi, \psi \in Form(\Phi, \Sigma)$, *and world terms* $t, s \in \mathbf{W}(\mathcal{F})$, *and in addition, the following labelled binary trees:*

Atomic actions $a_j \in \Sigma$:

$$(T\langle a_j\rangle) \qquad \begin{array}{c} T[\, t \Vdash \langle a_j\rangle\varphi \,] \\ | \\ T[\, \mathbf{F}_j(t) \Vdash \varphi \,] \end{array} \qquad\qquad (F\langle a_j\rangle) \qquad \begin{array}{c} F[\, t \Vdash \langle a_j\rangle\varphi \,] \\ | \\ F[\, \mathbf{F}_j(t) \Vdash \varphi \,] \end{array}$$

Composition $\alpha\beta$, *for actions* $\alpha, \beta \in Act(\Sigma)$:

$$(T\langle \alpha\beta\rangle) \qquad \begin{array}{c} T[\, t \Vdash \langle \alpha\beta\rangle\varphi \,] \\ | \\ T[\, t \Vdash \langle\alpha\rangle\langle\beta\rangle\varphi \,] \end{array} \qquad\qquad (F\langle \alpha\beta\rangle) \qquad \begin{array}{c} F[\, t \Vdash \langle \alpha\beta\rangle\varphi \,] \\ | \\ F[\, t \Vdash \langle\alpha\rangle\langle\beta\rangle\varphi \,] \end{array}$$

Sum $\alpha + \beta$, *for actions* $\alpha, \beta \in Act(\Sigma)$:

$$(T\langle \alpha+\beta\rangle) \qquad \begin{array}{ccc} & T[\, t \Vdash \langle \alpha+\beta\rangle\varphi \,] & \\ / & & \backslash \\ T[\, t \Vdash \langle\alpha\rangle\varphi \,] & & T[\, t \Vdash \langle\beta\rangle\varphi \,] \end{array}$$

$$(F\langle \alpha+\beta\rangle) \qquad \begin{array}{c} F[\, t \Vdash \langle \alpha+\beta\rangle\varphi \,] \\ | \\ F[\, t \Vdash \langle\alpha\rangle\varphi \,] \\ | \\ F[\, t \Vdash \langle\beta\rangle\varphi \,] \end{array}$$

Iteration α^*, *for an action* $\alpha \in Act(\Sigma)$:

$$(T\langle \alpha^*\rangle) \qquad \begin{array}{ccc} & T[\, t \Vdash \langle \alpha^*\rangle\varphi \,] & \\ / & & \backslash \\ T[\, t \Vdash \varphi \,] & & T[\, t \Vdash \langle\alpha\rangle\langle\alpha^*\rangle\varphi \,] \end{array}$$

$$(F\langle \alpha^*\rangle) \qquad \begin{array}{c} F[\, t \Vdash \langle \alpha^*\rangle\varphi \,] \\ | \\ F[\, t \Vdash \varphi \,] \\ | \\ F[\, t \Vdash \langle\alpha\rangle\langle\alpha^*\rangle\varphi \,] \end{array}$$

Definition 5.5.4 *The class of* **TPDL** tableaux *is defined inductively as follows:*

(i) *If τ is an atomic tableaux in which the world term t in the root entry is a world symbol $\mathbf{w}_i \in \mathbf{W}$, then τ is a* **TPDL** *tableaux.*

For the case $(F\Box)$, the condition that the $\mathbf{w}_j$ in $\mathbf{w}_i\mathbf{R}\mathbf{w}_j$ be "new" merely means that $j \neq i$; for definiteness, we may take $j = i + 1$.

For the case $(T\Box)$, the condition that $t\mathbf{R}s$ "occurs previously on this path" cannot be satisfied in this case, so an atomic tableaux τ with root entry $T[\, \mathbf{w}_i \Vdash \Box\varphi \,]$ consists of the root node only.

(ii) *If τ is a finite* **TPDL** *tableaux, P is a path in τ which does* not *contain contradictory entries:*

$$T[\, t \Vdash \varphi \,] \quad \text{and} \quad F[\, t \Vdash \varphi \,]$$

for any formula $\varphi \in Form(\Phi, \Sigma)$ and $t \in \mathbf{W}(\mathcal{F})$, and τ' is constructed from τ by extending P using one of the following construction rules, then τ' is a **TPDL** *tableaux.*

(Develop) *A signed forcing assertion E occurs on P and τ' is constructed from τ by appending an atomic tableaux with root entry E to the end of the path P.*

For the case $(F\Box)$, where E is of the form $F[\, t \Vdash \Box\varphi \,]$, the condition that the $\mathbf{w}_j$ in $t\mathbf{R}\mathbf{w}_j$ be "new" means that $j \in \mathbb{N}$ is the least integer such that $\mathbf{w}_j$ is yet to occur in any entry on τ.

For the case $(T\Box)$, where E is of the form $T[\, t \Vdash \Box\varphi \,]$, the condition that $t\mathbf{R}s$ "occurs previously on this path" means that $t\mathbf{R}s$ is an entry on P. If there are no entries $t\mathbf{R}s$ on P, for any $s \in \mathbf{W}(\mathcal{F})$, then as in (i)*, the atomic tableaux in this case consists only of the root node labelled E.*

(R-Reflex) *A world term $t \in \mathbf{W}(\mathcal{F})$ is relevant to some signed forcing assertion on P, and τ' is constructed from τ by adjoining to the end of P an entry $t\mathbf{R}t$.*

(R-Trans) *For some $t, s, r \in \mathbf{W}(\mathcal{F})$, accessibility assertions $t\mathbf{R}r$ and $r\mathbf{R}s$ both occur as entries on P, and τ' is constructed from τ by adjoining to the end of P the entry $t\mathbf{R}s$.*

($\mathbf{F}_j$-Cont) *For some $t, s \in \mathbf{W}(\mathcal{F})$, an accessibility assertion $t\mathbf{R}s$ occurs as an entry on P, $j \in \Sigma_\varphi$ where φ is the object formula in the root entry of P, and τ' is constructed from τ by adjoining to the end of* P *the entry $\mathbf{F}_j(t)\,\mathbf{R}\,\mathbf{F}_j(s)$.*

(iii) *If* $I \preceq \mathbb{N}$ *and* $\{\tau_n\}_{n\in I}$ *is a sequence of finite* **TPDL** *tableaux such that* τ_0 *is an atomic tableaux and for each* $n < \sup(I)$, τ_{n+1} *is constructed from* τ_n *by an application of clause* (ii), *then* $\tau = \bigcup_{n\in I}\tau_n$ *is a* **TPDL** *tableaux.*

Definition 5.5.5 *Given two signed forcing assertions* $S[\ t \Vdash \varphi\ ]$ *and* $S'[\ t' \Vdash \varphi'\ ]$, *with* $S, S' \in \{T, F\}$, $t, t' \in \mathbf{W}(\mathcal{F})$ *and* $\varphi, \varphi' \in Form(\Phi, \Sigma)$, *we say that* $S'[\ t' \Vdash \varphi'\ ]$ *is a* direct descendant *of* $S[\ t \Vdash \varphi\ ]$ *iff* $S'[\ t' \Vdash \varphi'\ ]$ *is an entry in the atomic tableaux which has* $S[\ t \Vdash \varphi\ ]$ *as its root entry.*

Lemma 5.5.6 *If* $S'[\ t' \Vdash \varphi'\ ]$ *is a direct descendant of* $S[\ t \Vdash \varphi\ ]$ *then* $\varphi' \prec \varphi$ *in the Fischer-Ladner ordering (Definition 5.4.5).*

Proof. Immediate from the definitions. ■

We will of course be using the Fischer-Ladner ordering rather than the subformula ordering in inductive proofs.

5.6 The Term Frame of a Path

The term frame is this setting is just a beefed-up version of the term frames for paths in **S4F** and **S4C** tableaux.

Definition 5.6.1 *Let* τ *be a* **TPDL** *tableaux with* $\varphi \in Form(\Phi, \Sigma)$ *the object of the root entry, and let* P *be a path through* τ. *We associate with* P *a unique Kripke frame* $\mathcal{K}_P = (W_P, R_P, \nu_P)$, *called the* term frame *for* P, *as follows.*

Let W_0 *be the set of all world symbols* $\mathbf{w}_i \in \mathbf{W}$ *that are the subject of a signed forcing assertion on* P. *Recall that* Σ_φ *is the set of indices of atomic actions appearing in* φ. *Define* $\Sigma_P = \Sigma_\varphi$, *and define:*

$$W_P \doteq \{\mathbf{F}_\sigma(\mathbf{w}_i) \mid \mathbf{w}_i \in W_0 \text{ and } \sigma \in \Sigma_P^*\}$$

i.e. $W_P \subseteq \mathbf{W}(\mathrm{F})$ *is the smallest subset of* $\mathbf{W}(\mathcal{F})$ *that contains all world terms that are the subject of some signed forcing assertion on* P *and is also closed under application of all appropriate* $\mathbf{F}_j$ *(for* $j \in \Sigma_\varphi$*); equivalently,* W_P *is the set of all terms in* $\mathbf{W}(\mathcal{F})$ *that are relevant to some signed forcing assertion on* P.

The relation R_P *on* W_P *is defined to be the reflexive, transitive and* $\mathbf{F}_j$*-functional closure, for all* $j \in \Sigma_P$, *of the relation* R *on* W_P *defined by:*

$$(t, s) \in R \quad \Leftrightarrow \quad t\mathbf{R}s \text{ is an entry on } P$$

for all $t, s \in W_P$. That is,

$$R_P \stackrel{\circ}{=} \bigcup_{m \in \mathrm{N}} R_m^+$$

where

$$\begin{array}{lcl} R_0^+ & = & \{(t,t) \mid t \in W_P\}, \textit{ the identity relation on } W_P \\ R_1^+ & = & R = \{(t,s) \in W_P \times W_P \mid t\mathbf{R}s \textit{ is an entry on } P\} \\ R_{m+1}^+ & = & R_m^+ \cup \{(t,s) \mid (\exists r \in W_P)\,(t,r) \in R_m^+ \textit{ and } (r,s) \in R_m^+\} \\ & & \cup\{(\mathbf{F}_j(t), \mathbf{F}_j(s)) \mid (t,s) \in R_m^+ \textit{ and } j \in \Sigma_\mathrm{P}\} \end{array}$$

For each $a_j \in \Sigma$, define the function $\nu_P(a_j) : W_P \to W_P$ by

$$(\nu_P(a_j))\,(t) = \begin{cases} \mathbf{F}_j(t) & \textit{if } j \in \Sigma_\mathrm{P} \\ t & \textit{otherwise} \end{cases}$$

for all $t \in W_P$; i.e. $\nu_P(a_j)$ is the term constructor *function $t \mapsto \mathbf{F}_j(t)$ on W_P if the atomic action a_j occurs in the object formula φ of the root entry of P, and the identity function otherwise.*

In this setting, the notion of a Kripke model $(\mathcal{K}, \eta)$ *agreeing with a path* P is the obvious extension of the notion in Definition 4.2.5: $\mathcal{K}$ is a quotient of $\mathcal{K}_\mathrm{P}$, under some quotient map $h : W_\mathrm{P} \to W$, where h preserves R_P and R; the family of maps $\{\nu(a)\}_{a \in \Sigma}$ for $\mathcal{K}$ and $\{\nu_\mathrm{P}(a)\}_{a \in \Sigma}$ for $\mathcal{K}_\mathrm{P}$ satisfies $\nu(a)(h(t)) = h(\nu_\mathrm{P}(a)(t))$; and h preserves the valuations described on P.

The process of constructing a **TPDL** tableaux proceeds analogously with the process of constructing an **S4F** or **S4C** tableaux. In particular, if P is a path through a **TPDL** tableaux τ, τ' is a **TPDL** tableaux obtained from τ by extending P by applying one of the tableaux construction rules in clause (ii) of Definition 5.5.4, and P$'$ is any path through τ' extending P, then:

(a) If the rule applied is *not* the $(F\Box)$ case of the (**Develop**) rule, then we have:

$$\mathcal{K}_{\mathrm{P}'} = \mathcal{K}_\mathrm{P}$$

(b) If the rule $(F\Box)$ is applied to an entry $F[\ t \Vdash \Box\psi\ ]$ occurring on P, where $t = \mathbf{F}_\sigma(\mathbf{w}_i) \in W_\mathrm{P}$, and $k \in \mathbb{N}$ is the least such that $\mathbf{w}_k$ is yet to occur in any entry on τ, then:

- $i < k$;
- $W_{\mathrm{P}'} = W_\mathrm{P} \cup \{\mathbf{F}_\omega(\mathbf{w}_k) \mid \omega \in \Sigma_{\mathrm{P}'}^*\}$, where $\Sigma_\mathrm{P} = \Sigma_{\mathrm{P}'}$;

- $R_{\mathsf{P}'}$ is the reflexive, transitive and $\mathbf{F}_j$-functional closure in $W_{\mathsf{P}'}$, for $j \in \Sigma_{\mathsf{P}}$, of $R_{\mathsf{P}} \cup \{(\mathbf{F}_\sigma(\mathbf{w}_i), \mathbf{w}_k)\}$;
- for each $j \in \Sigma_{\mathsf{P}}$, $\nu_{\mathsf{P}'}(a_j)$ is the term constructor function on $W_{\mathsf{P}'}$ uniquely extending $\nu_{\mathsf{P}}(a_j)$, and the identity function otherwise.

Theorem 5.6.2 Kripke Soundness of **TPDL** tableaux
For all formulas $\varphi \in Form(\Phi, \Sigma)$,
if **TPDL** $\vdash_T \varphi$ *then for all Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$, $\mathcal{K} \Vdash \varphi$.

Proof. The additional tableaux rules for complex actions create no complications in the main path extension lemma on the inductive construction of tableaux since these rules do not involve the introduction of new primitive world terms. ■

Contemplation of the tableaux construction rules reveals the following:

if $\mathbf{F}_\sigma(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{F}_\omega(\mathbf{w}_k)$ is an entry on P,
then $i < k$ and $\sigma = \pi\hat{\ }\omega$ for some (possibly empty) string $\pi \in \Sigma_{\mathsf{P}}^*$,
and $\mathbf{F}_\pi(\mathbf{w}_i)\,\mathbf{R}\,\mathbf{w}_k$ is an entry on P.

For strings $\sigma, \omega \in \Sigma_{\mathsf{P}}^*$, define the *final segment ordering* $\leq$ by: $\omega \leq \sigma$ iff $\sigma = \pi\hat{\ }\omega$ for some string $\pi \in \Sigma_{\mathsf{P}}^*$. Equipped with this extra machinery, we can easily describe the chains in the term frame.

Proposition 5.6.3 *Let* τ *be a* **TPDL** *tableaux, let* P *be a path through* τ, *and let* $\mathcal{K}_{\mathsf{P}} = (W_{\mathsf{P}}, R_{\mathsf{P}}, \nu_{\mathsf{P}})$ *be the term frame for* P.

Then for each $t = \mathbf{F}_\sigma(\mathbf{w}_i) \in W_{\mathsf{P}}$, *every* R_{P} *chain from* t *is of the form:*

$$\langle \mathbf{F}_\sigma(\mathbf{w}_i) \rangle * \langle \mathbf{F}_{\omega_j}(\mathbf{w}_{i_j}) \mid j \in J \rangle$$

for some $\emptyset \preceq J \preceq \mathbb{N}$, *where*

$$i < i_0 < i_j < i_{j+1} \quad \textit{and} \quad \omega_{j+1} \leq \omega_j \leq \omega_0 \leq \sigma$$

for all $j < \sup(J)$.

So the term frame $\mathcal{K}_{\mathsf{P}}$ *has the following properties:* W_{P} *is countably infinite,* R_{P} *is a partial order, and for each* $a_j \in \Sigma$, $\nu_{\mathsf{P}}(a_j)$ *is continuous and injective.*

Hence the induced continuous D-topological structure $\mathfrak{T}_{\mathsf{P}} = (W_{\mathsf{P}}, \mathcal{T}_{\mathsf{P}}, \nu_{\mathsf{P}})$ *is countable and* T_0, *with injective functions.*

Moreover, the basic open set for $t = \mathbf{F}_\sigma(\mathbf{w}_i) \in W_{\mathsf{P}}$ *in the cone topology* $\mathcal{T}_{\mathsf{P}} = \mathcal{T}_{R_{\mathsf{P}}}$ *is of the form:*

$$B_t = \{\mathbf{F}_\sigma(\mathbf{w}_i)\} \cup \{\mathbf{F}_{\omega_n}(\mathbf{w}_{i_n}) \mid n \in N\}$$

for some (possibly empty) subset $N \subseteq \mathbb{N}$, *where* $i < i_n < i_{n'}$ *and* $\omega_n \leq \sigma$ *for all* $n, n' \in N$ *with* $n < n'$.

Proof. The obvious modification of the proof of Proposition 4.2.8. ■

5.7 Completeness of Tableaux

Extending the completeness result for the bimodal logics to **TPDL** is quite straightforward, once one has in place the Fischer-Ladner order for inductions.

Given a tableaux with $\varphi \in Form(\Phi, \Sigma)$ as the object of the root entry, we can fix an ordering on terms $s_j = \mathbf{F}_\sigma(\mathrm{w}_i) \in \mathbf{W}(\mathcal{F})$ such that $\sigma \in \Sigma^*_\varphi$ and $i \in \mathbb{N}$, by taking some well-ordering on $\mathbb{N} \times \Sigma^*_\varphi$. The definition of the m^{th} occurrence e of a signed forcing assertion E being *reduced* on a path P, is carried over from Definition 4.4.1; the $m = p(j, l)$ is used only in the $(T\Box)$ case, when the term s_j is dealt with for the l^{th} time. The definition of a tableaux being *finished* is also carried over.

Definition 5.7.1 *For each formula $\varphi \in Form(\Phi, \Sigma)$, we define the* complete systematic **TPDL** tableaux τ^φ *for φ to be:*

$$\tau^\varphi = \bigcup_{n \in I} \tau_n$$

for some $I \preceq N$, where the sequence of finite **TPDL** *tableaux $\{\tau_n\}_{n \in I}$ for φ is defined inductively as in Definition 4.4.3, except that **Case 3** of stage $n+1$, which is modified as follows:*

***Case 3**: $n = 4k + 3$, for $k \in \mathbb{N}$.*

Construct τ_{n+1} from τ_n by appending, to the end of each non-contradictory path P through τ_n, an entry $\mathbf{F}_m(s_i)\,\mathbf{R}\,\mathbf{F}_m(s_j)$ for the least $i \leq k$, the least $j \leq k$ and the least $m \leq k$, such that:

- $m \in \Sigma_\varphi$,
- *$s_i \mathbf{R} s_j$ is an entry on P, and*
- *$\mathbf{F}_m(s_i)\,\mathbf{R}\,\mathbf{F}_m(s_j)$ does not yet occur on P.*

Proposition 5.7.2 *For each formula $\varphi \in Form(\Phi, \Sigma)$, the* **TPDL**-*CST* τ^φ *is finished.* ■

Theorem 5.7.3 *If $\tau = \tau^{\varphi} = \bigcup_{n\in I}\tau_n$ is the* **TPDL**-*CST for $\varphi \in Form(\Phi, \Sigma)$, P is a non-contradictory path through τ^{φ}, $\mathcal{K}_P = (W_P, R_P, \nu_P)$ is the term frame of P, and η_P is the path valuation for $\mathcal{K}_P$ (Definition 4.2.4), then*

(i) *for all $t, s \in W_P$,*

$$t\mathbf{R}s \text{ is an entry on P} \quad \Leftrightarrow \quad (t,s) \in R_P$$

(ii) *for all $t \in W_P$ and all $\psi \in FL(\varphi)$,*

$$\begin{array}{lcl} T[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \Vdash^{\mathcal{K}_P}_{\eta_P} \psi \\ F[\, t \Vdash \psi \,] \text{ is an entry on P} & \Rightarrow & t \nVdash^{\mathcal{K}_P}_{\eta_P} \psi \end{array}$$

Hence the identity function on W_P witnesses that $(\mathcal{K}_P, \eta_P)$ agrees with P.

Proof. The proof of (i) requires only a cosmetic change to the argument, verifying that P is closed under the ($\mathbf{F}_j$-**Cont**) rule for each $j \in \Sigma_\varphi$. For (ii), there are extra cases in the induction, and the induction proceeds with respect to the Fischer-Ladner order $\prec$.

For example, for $T\langle\alpha\beta\rangle$, assume by induction that the result holds for all formulas $\prec \langle\alpha\beta\rangle\psi$ and all world terms in W_P. Fix $t \in W_P$ and suppose $T[\, t \Vdash \langle\alpha\beta\rangle\psi \,]$ is an entry on P. Then since (every occurrence of) this entry is reduced, $T[\, t \Vdash \langle\alpha\rangle\langle\beta\rangle\psi \,]$ is an entry on P. Hence by the induction hypothesis, we have $t \Vdash_{\eta_P} \langle\alpha\rangle\langle\beta\rangle\psi$. Since

$$\langle\alpha\beta\rangle\psi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\psi$$

is Kripke valid, it follows that $t \Vdash_{\eta_P} \langle\alpha\beta\rangle\psi$, as required.

The other additional cases proceed similarly, appealing to the Kripke validity of:

$$\begin{array}{l} \langle\alpha+\beta\rangle\psi \leftrightarrow \langle\alpha\rangle\psi \vee \langle\beta\rangle\psi \\ \langle\alpha^*\rangle\psi \leftrightarrow (\psi \vee \langle\alpha\rangle\langle\alpha^*\rangle\psi \quad \blacksquare \end{array}$$

Theorem 5.7.4 Kripke completeness of **TPDL** tableaux

For each formula $\varphi \in Form(\Phi, \Sigma)$, if **TPDL** $\nvdash_T \varphi$, *then there is a countable partially-ordered Kripke frame $\mathcal{K}$ for $\mathcal{L}_\Box(\Phi, \Sigma)$ and a valuation η for $\mathcal{K}$ such that $(\mathcal{K}, \eta) \nVdash \varphi$.*

Proof. Same as the proof of Theorem 4.4.8. ■

5.8 Finite Quotients and Decidability

We have in fact already proved the finite model property in Proposition 5.4.7, but since tableaux are more pleasing to contemplate that maximal consistent sets, we sketch the finite quotient argument for the term frame. If one were inspired, one could formalize the correspondence between the filtration of the non-standard canonical model through the Fischer-Ladner closure, and the finite quotient constructed here.

Definition 5.8.1 *For each $\varphi \in Form(\Phi, \Sigma)$, let $FL(\varphi)$ denote the Fischer-Ladner closure of φ, as defined in Definition 5.4.5.*

Define signed Fischer-Ladner closure *of φ, $SFL(\varphi)$, by:*

$$SFL(\varphi) \doteq \{ T[\psi] \mid \psi \in FL(\varphi)\} \cup \{ F[\psi] \mid \psi \in FL(\varphi)\}$$

Let τ be a **TPDL**-*CST for $\varphi \in Form(\Phi, \Sigma)$.*

For each world term $t \in \mathbf{W}(\mathcal{F})$ and path P through τ, define

$$\begin{array}{rcl} S_{\mathtt{P}}(t) & \doteq & \{ T[\psi] \mid T[\, t \Vdash \psi \,] \text{ is an entry on P} \} \\ & \cup & \{ F[\psi] \mid F[\, t \Vdash \psi \,] \text{ is an entry on P} \} \end{array}$$

A subset $S \subseteq SFL(\varphi)$ is called inconsistent *if there is a $\psi \in FL(\varphi)$ such that both $T[\psi] \in S$ and $F[\psi] \in S$; and* consistent *otherwise.*

On a path P through a tableaux, every set $S_{\mathtt{P}}(t) \subseteq SFL(\varphi)$, and if the cardinality $|FL(\varphi)| = n$, then the total number of consistent subsets $S \subseteq SFL(\varphi)$ is:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Definition 5.8.2 *Let τ be the* **TPDL**-*CST for $\varphi \in Form(\Phi, \Sigma)$, let P be a path through τ, and let $\mathcal{K}_P = (W_P, R_P, \nu_P)$ be the term frame for P, with η_P the path valuation.*

Define an equivalence relation $\equiv_{\mathtt{P}}$ on W_P by:

$$t \equiv_{\mathtt{P}} s \quad \textit{iff} \quad S_{\mathtt{P}}(t) = S_{\mathtt{P}}(s)$$

Let $\bar{t} = \{s \in W_P \mid t \equiv_{\mathtt{P}} s\}$ and let $W_P^{\#}$ denote the set of all $\equiv_{\mathtt{P}}$-equivalence classes $\bar{t}$. Define $\mathcal{K}_P^{\#} = (W_P^{\#}, R_P^{\#}, F_P^{\#})$ to be the minimal quotient under the surjective map $h : W_P \to W_P^{\#}$ given by $h(t) = \bar{t}$; i.e. for all $t, s \in W_P$,

$$h(t) = h(s) \quad \Leftrightarrow \quad S_{\mathtt{P}}(t) = S_{\mathtt{P}}(s)$$

and

$$(h(t), h(s)) \in R_P^{\#} \quad \Leftrightarrow \quad (t, s) \in R_P$$

and

$$\left(\nu_P^{\#}(a_j)\right)(h(t)) = h\left((\nu_P(a_j))(t)\right) = \begin{cases} h(\mathbf{F}_j(t)) & \textit{if } j \in \Sigma_\varphi \\ h(t) & \textit{otherwise} \end{cases}$$

Theorem 5.8.3 Finite model property for **TPDL**.

Let τ be the **TPDL**-*CST for $\varphi \in Form(\Phi, \Sigma)$ with root $F[\, \mathbf{w}_0 \Vdash \varphi \,]$, and suppose P is a non-contradictory path through τ. Let $\mathcal{K}_P = (W_P, R_P, \nu_P)$ be the term frame for P, with η_P its canonical valuation, and let $\mathcal{K}_P^{\#} = (W_P^{\#}, R_P^{\#}, \nu_P^{\#})$ be the minimal quotient of $\mathcal{K}_P$ under $\equiv_P$. Then:*

(a) *The equivalence relation $\equiv_P$ is of finite index: if $n = |FL(\varphi)|$ then $\left|W_P^{\#}\right| \leq 3^n$.*

(b) *The induced valuation $\eta_P^{\#} : W_P^{\#} \to \mathcal{P}(\Phi)$ given by:*

$$\eta_P^{\#}(h(t)) = \eta_P(t) = \{p \in \Phi \mid T[\, t \Vdash p \,] \;\; \textit{is an entry on } \mathrm{P}\}$$

is well-defined.

(c) *For all $\psi \in FL(\varphi)$ and $t \in W_P$,*

$$h(t) \Vdash_{\#} \psi \quad \Leftrightarrow \quad t \Vdash \psi$$

where $\Vdash$ abbreviates $\Vdash_{\eta_P}^{\mathcal{K}_P}$ and $\Vdash_{\#}$ abbreviates $\Vdash_{\eta_P^{\#}}^{\mathcal{K}_P^{\#}}$.

Hence $(\mathcal{K}_P^{\#}, \eta_P^{\#})$ agrees with P, and in particular, $h(\mathbf{w}_0) \nVdash_{\#} \varphi$, so

$$(\mathcal{K}_P^{\#}, \eta_P^{\#}) \nVdash \varphi$$

Proof. Using the Fischer Ladner order, the induction in part (c) requires only straightforward appeals to the definitions of forcing. ■

Corollary 5.8.4 *The logic* **TPDL** *is decidable.*

Corollary 5.8.5 *For each formula $\varphi \in Form(\Phi, \Sigma)$, the following are equivalent:*

(1.) **TPDL** $\vdash_T \varphi$

(2.) **TPDL** $\vdash_H \varphi$

(3.) $\mathfrak{T} \models \varphi$ *for all topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$,

(4.) $\mathfrak{T} \models \varphi$ *for all D-topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$,

(5.) $\mathcal{K} \Vdash \varphi$ *for all Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$,

(6.) $\mathfrak{T} \models \varphi$ *for all countable* T_0 *D-topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$,

(7.) $\mathcal{K} \Vdash \varphi$ *for all countable partially-ordered Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$,

(8.) $\mathcal{K} \Vdash \varphi$ *for all finite Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_\Box(\Phi, \Sigma)$.

Proof. (8.) $\Rightarrow$ (1.) is the finite model property, and (5.) $\Rightarrow$ (8.) is trivial. (7.) $\Rightarrow$ (1.) is the completeness theorems for **TPDL** tableaux, in Theorem 5.7.4. (1.) $\Rightarrow$ (5.) is the Kripke soundness of **TPDL** tableaux, in Theorem 5.6.2. (3.) $\Rightarrow$ (4.) and (5.) $\Rightarrow$ (7.) are trivial. (4.) $\Leftrightarrow$ (5.) and (6.) $\Leftrightarrow$ (7.) are Corollaries 5.2.6 and 5.2.7. (2.) $\Rightarrow$ (3.) is the topological soundness of the Hilbert-style proof system, in Proposition 5.4.2 (3.2.2). And (5.) $\Rightarrow$ (2.) is the Kripke completeness results for the Hilbert-style proof system, in Proposition 5.4.7. ■

5.9 Conclusion

The larger goal of this investigation was to provide a logical foundation for hybrid control systems in which *topological* structure is taken seriously. This work is at least a modest contribution to that endeavor, and will serve as a base for future research.

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